Stable Market Segmentation against Price Discrimination*

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Abstract

According to current data regulations, consumers are mobile among different markets, which endogenizes market segmentation. Considering such strategic interactions, we say that a market segmentation is stable if no group of consumers has an incentive to deviate. We show that in every stable market segmentation, the producer surplus remains at the uniform monopoly level, and the consumer surplus takes a value between the buyer-optimal level and the uniform monopoly level. Remarkably, no consumer is worse off than in the case of uniform monopoly. Therefore, our results justify the Pareto optimum of price discrimination and reveal the welfare implications of current regulations.

Keywords: Price discrimination, Market segmentation, Right to Be Forgotten, Strategic consumer, Welfare;

JEL Classification: D42, D83, L12

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1 Introduction

Consumers online, let us stand up and protect ourselves from price discrimination! The proliferation of data technologies enables online sellers to collect vast amounts of consumer information during transactions. By data inference, sellers can accurately deduce consumer preferences and generate tags that denote various consumer features. These tags prove valuable in predicting consumers' potential needs and willingness to pay. While sellers tout the benefits of these tags in enhancing services, those with market power can exploit these tags to segment the market and engage in price discrimination. This practice can be particularly harmful to consumers when sellers hold a monopoly. Consequently, scholars and competition authorities have proposed the prohibition of price discrimination as a means to safeguard consumers.[1]

Recent advancements in privacy protection have prompted a novel approach to regulating market power that surpasses traditional methods. This approach revolves around empowering consumers to strategically select their market segments. To uphold consumers' right to be forgotten, the European Union (*General Data Protection Regulation*) and China (*Internet Information Service Algorithmic Recommendation Management Provisions*) have instituted regulations allowing consumers to remove their labels.[2] In a more radical move, an exposure draft in China even allows consumers to freely modify their tags.[3] In addition to ongoing regulatory debates, real-time free editing is already a reality. For example, users on Instagram, Pinterest, or Tiktok can change the channels they join or accounts they follow, impacting the prices of products offered in the embedded online shops. Clearly, rational consumers will navigate between segments in search of better prices, thereby potentially shielding themselves from price discrimination.

While data regulations equip consumers with a tool for self-protection, are these measures sufficient? Considering consumers' decentralized, self-interested, and insignificant nature, can they be effectively protected against a monopolist? To address this question, we build a model to explore the welfare implications when consumers can freely choose

[1]For example, in July 2021, the State Administration for Market Regulation of China released an exposure draft of the *Provisions on Administrative Penalties for Price Violations*, which prohibits price discrimination in Article 4. It is the latest update since 2010.

[2]All iOS apps on the App Store are required to offer an in-app account deletion option by July 30, 2022 (see https://developer.apple.com/support/offering-account-deletion-in-your-app). Moreover, all iOS apps should let people use them without a login (see App Store Review Guideline 5.1.1(v) in https://developer.apple.com/app-store/review/guidelines/#data-collection-and-storage).

[3]When the Chinese government drew up its data regulations in 2021, free editing was allowed in the draft version, indicating that different tag-based markets would be completely circulating. However, this right was removed when the regulation was promulgated in 2022, remaining the right to erase tags only.

their tags, known as identity management (Chen, Choe and Matsushima, 2020). Tags are assumed to be arbitrarily editable for four reasons: (i) it is an existing proposal in China; (ii) it is an acceptable approximation to the partial erasure mandated by the *Right to Be Forgotten*;[4] (iii) it is in line with the fact that tags now describe consumers' taste, which would not violate any restriction about providing fake information;[5] (iv) more importantly, it is a theoretical limit case with complete freedom as consumers' data protection is improving. Initially, the seller can charge different prices in each tag-based market. However, if tags are easily modified, consumers may find it advantageous to deviate from the current market by manipulating their tags to obtain a more favorable price.

To gain insights into consumers' strategic behavior in online markets, we introduce the concept of *stable* market segmentation to lay the decentralized foundation for market segmentation.[6] Given that consumers often communicate with their friends/relatives/neighbors in social networks and through online review systems or chat groups before making their market choices[7], we define a market segmentation as stable if no group of consumers has an incentive to deviate. In this paper, our primary objective is to study the existence and potential welfare consequences of stable market segmentation. Specifically, do consumers benefit from a stable market segmentation compared to a scenario where they cannot edit their tags? Can the monopolist obtain a higher profit than in a uniform monopoly? Is every stable segmentation considered Pareto-improving over the outcome of a uniform monopoly?

Now we describe our model in detail. We consider a monopolistic producer who sells a homogeneous product to a continuum of consumers, each having a unit demand. The

[4]The Right to Be Forgotten allows consumers to refuse discriminatory pricing by removing all labels. However, consumers may act more strategically by deleting specific dimensions of labels or only part of purchase history. Additionally, consumers may create additional accounts or invite family members and friends to purchase the item in their accounts to obtain lower prices. This suggests that different markets are circulating rather than a simple binary choice between original and anonymous markets.

[5]Since tags used in online systems are often subjective rather than objective, it is impossible to verify the authenticity of tags. For example, some consumers may be tagged because they like products in red. He can change the tag to indicate that he loves green now. This change cannot be punished if we forbid fake information.

[6]In contrast with studies wherein the producer passively accepts the segmentation or those focusing on the segmentation decision made by the central planner (see Bergemann, Brooks and Morris (2015) for an example), our framework provides a new perspective by studying consumer decisions, which is essential but ignored in online markets. Moreover, Peivandi and Vohra (2021) examines the possibility of segmentation induced by competing mechanisms.

[7]When a consumer finds an arbitrage opportunity, he can tell his friends/relatives/neighbors in his social network. Hence, a group of consumers would deviate together. Moreover, customers often communicate with each other using Amazon's product review system before purchasing. Similarly, millions of WeChat chat groups exist to exploit loopholes to acquire various goods at incredibly low prices. Data cooperatives, such as Data Commons, are another example of consumers coordinating with each other.

producer offers different take-it-or-leave-it prices in each market, while consumers have their reservation prices. The game begins with a tag editing process, in which consumers simultaneously select their tags to form tag-based segmentation. Equivalently, tags can be assigned by the producer or a third party such that consumers are incentive-compatible. Once the segmentation is realized, tag editing is shut down, and the producer observes the value distribution and chooses the minimum optimal price in each market, breaking ties in favor of consumers. Subsequently, consumers make their purchasing decisions based on the prevailing prices. We assume that consumers are rational in forecasting the prices for any market segmentation. A segmentation is stable in the tag-editing stage if no group of consumers can all derive strictly higher utility by modifying their tags. Here, a modification can change market prices since sellers online typically adopt algorithmic pricing, and consumers are rational in realizing this fact.

We begin by demonstrating the existence of stable segmentation since no segmentation at all is always stable. Any deviation must create a new market, which is not profitable for the consumer with the lowest valuation within the deviation group. We then characterize the welfare consequences of all stable segmentation. We show that producer surplus remains fixed at the uniform monopoly profit. Meanwhile, consumer surplus can range from the uniform monopoly surplus to the maximum consumer surplus attainable under arbitrary segmentation (also known as the *buyer-optimal* outcome). Our results can be shown by construction, as explained below.

Besides the uniform monopoly outcome, we also know that the buyer-optimal outcome is achievable. To implement this extreme outcome, we can iteratively partition the aggregate market into several *extremal markets*, where every valuation on the support is optimal (Bergemann, Brooks and Morris, 2015). The producer merely charges the minimum valuation in each market. Prices in extremal markets are sensitive to entry, which enables stable market segmentation to have different prices for different markets.

With the above two attainable outcomes, we cannot immediately conclude that any outcome in between is feasible since a convex combination of stable market segmentation may not necessarily be stable. However, we can construct a family of stable segmentation to circumvent the difficulties. In particular, we start with the segmentation that implements the buyer-optimal outcome mentioned above. We then merge markets with the highest prices to create a new market with a share equal to a parameter between zero and one. Assuming that the critical market can be proportionally divided, the consumer surplus in the construction is continuous in the parameter. Moreover, two extreme cases of this construction correspond to the uniform monopoly and buyer-optimal outcomes, respectively. Thus, by continuity, any consumer surplus in between can be obtained.

Our analysis also reveals that prices in any stable segmentation cannot exceed the uniform monopoly price. This result implies that price discrimination is Pareto-improving compared to the uniform monopoly. Traditional economic theory suggests that third-degree pricing discrimination has mixed effects on consumers, with some benefiting and others losing compared to the uniform monopoly. Our results challenge this view and show that, when consumers are armed with a tag-editable policy, third-degree price discrimination can benefit all consumers. These insights defend against the criticism that price discrimination by a monopolist harms consumers (at least in part).

We further provide necessary and sufficient conditions for a segmentation to be stable, which we consider an important technical contribution. Since there are infinite potential deviation possibilities, we cannot validate a stable segmentation even with a simple structure. For example, halving the aggregate market into two identical markets may result in an unstable segmentation. Thus, a simplified condition for stability is extraordinarily important:[8] If a segmentation is unstable, an inflow market must exist such that some consumers from other markets find it better to enter this market jointly.[9] The stable condition is thus excluding all such possibilities through linear programming.

Although previous studies have acknowledged the buyer-optimal outcome,[10] a key question remains how can it be implemented without an omniscient central planner? We show that social-optimal stable segmentations not only exist but are also buyer-optimal. Therefore, our analysis clarifies that empowering decentralized consumers to edit their tags can implement the centralized optimal solution. Moreover, we raise a geometrical method that fully characterizes every segmentation that satisfies both stability and social/consumer optimum, which is also a technical contribution.

Our findings have important implications for E-commerce, where data trust or online platforms act as mediators of individual data (MID)[11] or market designers for bilateral trade, caring about both producer and consumer surplus. As stable market segmentation

[8] Finding feasible collusion is extremely hard (NP-hard in the theory of computational complexity) in many situations, such as voting (Davies et al., 2014; Walsh, 2011).

[9] This simplified condition exhibits a polynomial time verification for stability or finding incentive-compatible group manipulations (if it exists).

[10] See Bergemann, Brooks and Morris (2015); Roesler and Szentes (2017) for examples.

[11] For more discussion about MID, see *A Blueprint for a Better Digital Society* by Jaron Lanier and E. Glen Weyl in `https://hbr.org/2018/09/a-blueprint-for-a-better-digital-society`.

is generally not unique, the mediator may choose the desirable segmentation for a broader business model or social responsibility. The desirable market outcome is uniquely pinned down at the buyer-optimal outcome due to legally mandated tag editing. However, if tag editing is not enforced, multiple social-optimal welfare consequences (other than the buyer-optimal outcome) can be considered by the mediator (Bergemann, Brooks and Morris, 2015). Hence, tag editing protects consumers from the mediator. Our paper provides algorithms that enumerate each socially optimal stable segmentation, allowing the mediator to easily segment the market according to our method.

Finally, we draw attention to the similarities and differences between tag editing and ex-post arbitrage. Both mechanisms can prevent monopolistic exploitation of consumers.[12] However, for regulators or market designers, tag editing is more efficient than facilitating post-purchase arbitrage in the presence of third-degree price discrimination. This is because, in the case of frictionless arbitrage, the market outcome boils down to the uniform monopoly, restricting consumer surplus at the uniform monopoly level. In contrast, tag editing can favor consumers without hurting the monopolist.

Related Literature

Our work springs from two strands of literature. First, our paper primarily belongs to the literature studying the welfare consequences of price discrimination (e.g., Aguirre, Cowan and Vickers (2010) and Cowan (2016)). In a seminal paper, Bergemann, Brooks and Morris (2015) characterizes all possible welfare consequences in third-price discrimination with exogenous segmentation. The shaded triangle in Figure 1 depicts all available surplus pairs, in which A marks the uniform monopoly outcome and C marks the buyer-optimal segmentation.[13] Using their terminology, we show that any possible consequence lies between A and C if the market segmentation is endogenized by decentralized consumers. Additionally, Ichihashi and Smolin (2023) investigate how the surplus triangle shrinks when more real-world restrictions are added in Bergemann, Brooks and Morris (2015). Yet, they assume that the producer has private information unknown to the central designer, which differs from ours.

The market segmentation problem analyzed in Bergemann, Brooks and Morris (2015)

[12] If consumers myopically recognize that prices will not change (instead of updating prices fully rationally) after a deviation, only the uniform monopoly outcome is stable. This coincides with the situation allowing ex-post arbitrages but forbidding consumers' circulation.

[13] Point B marks the first-degree price discrimination outcome, in which consumers with the same valuations are grouped. Point D marks the outcome where social welfare is minimized.

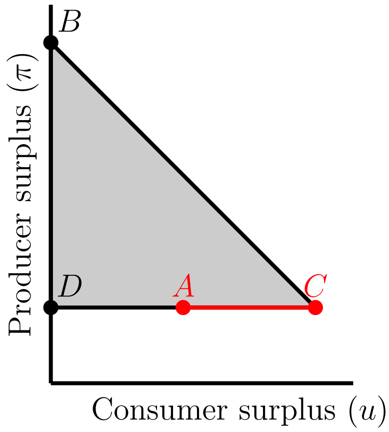

Figure 1: Surplus Triangle

can also be viewed as an information design problem with a single consumer. Then, Roesler and Szentes (2017) study the problem where the buyer has a prior value distribution and designs a signal structure to learn the distribution. Condorelli and Szentes (2020) further endogenize the selection of value distribution. However, these studies are limited in their scope to a single seller and a homogeneous product.[14] To complement these studies, we incorporate multiple strategic consumers, endogenizing the market segmentation decision, which has not been explored in the literature to our knowledge.

Second, this paper is also linked to the literature on strategic consumers. One mainstream of literature assumes repeated purchases from a monopolist. In such *behavior-based price discrimination* scenarios, the monopolist obtains information about consumers through their purchase history, and consumers act myopically or strategically against the monopolist (Fudenberg and Tirole, 2000; Gehrig and Stenbacka, 2007; Shen and Villas-Boas, 2018; Bonatti and Cisternas, 2020).[15] Acquisti and Varian (2005) argues that purchase history may harm consumers in later periods, forcing them to protect their privacy. Chen and Zhang (2009) consider strategic consumers seeking the lowest price when facing dynamic pricing. Our paper differs structurally from those studies, as strategic consumers in our model make only a single purchase decision, while their strategic feature arises from their interaction with other consumers. In contrast, consumers in behavior-based price discrimination are strategic for their forward-looking manners.

The strategic concerns of consumers are also present in their voluntary disclosure de-

[14]See Ichihashi (2020); Deb and Roesler (2021); Haghpanah and Siegel (2023*a*, 2022) for examples of bilateral trade models with multiple products; and Armstrong and Zhou (2022); Belleflamme, Lam and Vergote (2020); Elliott et al. (2021); Chen, Choe and Matsushima (2020) for examples with multiple (mostly two) strategic and competitive producers.

[15]See the detailed survey conducted by Fudenberg and Villas-Boas (2006).

cisions before the purchase. Sher and Vohra (2015) consider that each consumer privately belongs to multiple segments. However, they can credibly disclose their segment, limiting the monopolist's power. Ali, Lewis and Vasserman (2023) investigate the incentive to voluntarily disclose hard evidence (exact information or range about personal preferences), which builds the market segmentation. Hidir and Vellodi (2021) study the soft information about consumers' valuations and introduce the incentive-compatible market segmentation. In their settings, the strategic side of consumers is manifested by their decisions on their data, which indirectly influences market segmentation. While in ours, each consumer directly chooses the market segment.

In a cooperative setting, Haghpanah and Siegel (2023*b*) study the formation of market segmentation by consumers. As a result, the market they analyzed must be efficient, the producer surplus may arise, and the outcome may not be Pareto-improving compared with the aggregate market. In contrast, the market segmentations analyzed in our paper may be inefficient, the producer surplus is fixed, and the outcome is always Pareto-improving.[16]

The rest of the paper is organized as follows. Section 2 sets up a general framework for tag-editable market segmentation. Section 3 conducts analysis on stable market segmentations. Section 4 further explores those stable and social-optimal segmentations. In Section 5, we illustrate several possible extensions and discuss the main implications of our results. Section 6 concludes the paper. All technical proofs are relegated to the Appendix.

2 Model

We establish the basic model of market segmentation in Section 2.1 and introduce the tag-editable framework in Section 2.2.

2.1 Basic Setup and Market Segmentation

A monopolistic producer sells a homogeneous product to a continuum of consumers. Without loss of generality, the total mass of the consumers is normalized to one, and the constant marginal cost is normalized to zero. The consumers' willingness-to-pay for the product can take K positive values in $V \triangleq \{v_1, v_2, \cdots, v_K\}$ with $v_{k-1} < v_k$.

A *market* $\mathbf{x}$ is a K-dimensional vector, where x_k denotes the mass of consumers with

[16] Haghpanah and Siegel (2023*b*) and our paper independently adopt the name "stable market segmentation" in February 2022. Despite the similar name, the two notions of stability have large differences.

value v_k. Since each consumer will purchase the good if his valuation is greater than or equal to the price, $\sum_{i=k}^{K} x_i$ is the demand at any price in the interval $(v_{k-1}, v_k]$. Pricing v_k is *optimal* in $\mathbf{x}$ if $v_k \sum_{i=k}^{K} x_i \geq v_j \sum_{i=j}^{K} x_i$ for all $j \in \{1, 2, \cdots, K\}$.

There is an aggregate market $\mathbf{x}^*$ with $\|\mathbf{x}^*\|_1 = 1$. A market segmentation $\sigma(\mathbf{x}^*)=\{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$ divides the aggregate market $\mathbf{x}^*$ into finite markets such that $\sum_{i=1}^{t} \mathbf{x}_i = \mathbf{x}^*$. Since markets within a segmentation can be identical, σ may contain repeated elements. The set of possible segmentations for $\mathbf{x}^*$ is denoted by $\{\sigma \big| |\sigma| < \infty, \sum_{\mathbf{x}_i \in \sigma(\mathbf{x}^*)} \mathbf{x}_i = \mathbf{x}^*\}$.

The producer offers a take-it-or-leave-it price to maximize the profit in each market. The deterministic pricing strategy, denoted by ϕ, maps every market to a non-negative price. If multiple prices are optimal, selecting the minimum breaks the tie by maximizing consumer surplus. This rule, denoted by $\phi^{\min}$, is called *minimum optimal pricing rule.*[17]

Assumption 1 (Pricing algorithm)**.** *The producer adopts the minimum optimal pricing.*

Let u, π, and w denote the consumer surplus, producer surplus, and total surplus, respectively. For aggregate market, we denote $v^* = v_{i^*} = \phi^{\min}(\mathbf{x}^*)$ as the minimum optimal price. Under the uniform monopoly, producer surplus is $\pi^* = v^* \sum_{j=i^*}^{K} x_j^*$ and consumer surplus is $u^* = \sum_{j=i^*}^{K} (v_j - v^*) x_j^*$. The maximum feasible social welfare is $\overline{w} = \sum_{i=1}^{K} v_i x_i^*$, in which all consumers purchase the good. Under the segmentation $\sigma(\mathbf{x}^*)$, producer surplus is $\sum_{\mathbf{x} \in \sigma} \phi^{\min}(\mathbf{x}) \sum_{j: v_j \geq \phi^{\min}(\mathbf{x})} x_j$; consumer surplus is $\sum_{\mathbf{x} \in \sigma} \sum_{j: v_j \geq \phi^{\min}(\mathbf{x})} (v_j - \phi^{\min}(\mathbf{x})) x_j$; and the social welfare is $\sum_{\mathbf{x} \in \sigma} \sum_{j: v_j \geq \phi^{\min}(\mathbf{x})} v_j x_j$. All possible surplus pairs (u, π) that are attainable by some segmentation are summarized in Lemma 1(i), which is often called the *surplus triangle*.

Lemma 1. *(i)* (Bergemann, Brooks and Morris, 2015) *There exists a segmentation and associated pricing rule ϕ to implement (u, π) if and only if $u \geq 0, \pi \geq \pi^*$, and $u + \pi \leq \overline{w}$. (ii) Given $\phi^{\min}$, outcomes with $u \in [0, u^*)$ and $\pi = \pi^*$ are no longer attainable.*

Lemma 1(ii) suggests that fixing the pricing rule at $\phi^{\min}$ has a minor influence on welfare consequences for the general market segmentation problem.[18] Hence, Assumption 1 is considered an acceptable assumption, provided that a deterministic pricing strategy is adopted to avoid ambiguity in consumers' beliefs on market prices.

[17] Barreto, Ghersengorin and Augias (2022) also assume that the monopolist charges at each market the smallest one among the optimal prices in that market. By contrast, some works (e.g., Bergemann, Brooks and Morris (2015)) allow the pricing rule to be randomized and break tie arbitrarily, which makes strategic consumers unable to form unambiguous beliefs on pricing.

[18] Compared to the surplus triangle, the set of unattainable outcomes caused by the minimum optimal pricing rule has measure zero.

2.2 Tag-editable Framework

In the remainder of this section, we model market segmentation with the presence of strategic consumers. The introduction of strategic consumers to the price discrimination framework is inspired by the manipulable tag system online. Unlike the conventional literature (Bergemann, Brooks and Morris, 2015), the market partition in our tag-editable market segmentation framework is driven by consumers' incentives.

2.2.1 Game and Timeline

Our game involves a tag-editing stage and an implementation stage. In the first stage, consumers simultaneously choose their tags to form a tag-based market segmentation. Alternatively, we can assume that tags are assigned by the producer or a third party such that consumers are incentive-compatible with the assignment.

In the second stage, tags are not editable. The producer sets prices in all markets to implement third-degree price discrimination, which, in reality, is often realized *automatically* by the pricing *algorithm*. Meanwhile, each consumer purchases the good if and only if his reservation price is no less than the price. Figure 2 summarizes our timeline.

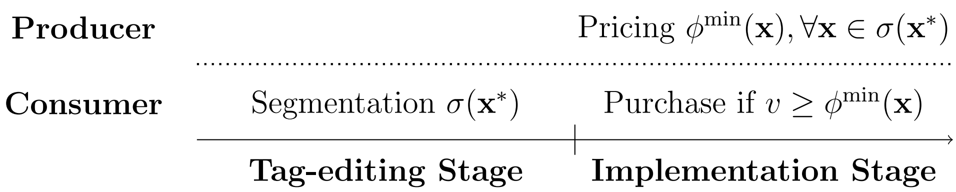

Figure 2: Timeline

Given the algorithmic pricing rule, consumers act strategically when editing their tags. To this end, we introduce the concept of *stable* market segmentation in the following.

2.2.2 Stable Market Segmentation

In defining stable segmentation, consumers can form a group and change their tags collaboratively. A group of consumers is denoted by $\mathbf{y} = \sum_{i=1}^{t} \mathbf{y}_i$, where t is the number of markets, $\mathbf{y}_i$ collects consumers from market $\mathbf{x}_i$ who want to deviate, and $\mathbf{0} \leq \mathbf{y}_i \leq \mathbf{x}_i$. Notice that the consumer with value v_k in market $\mathbf{x}$ has the utility $\max\left\{v_k - \phi^{\min}(\mathbf{x}), 0\right\}$.

Definition 1 (Stable)**.** *A segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$ is* ***stable****, if for any group of consumers $\mathbf{y}$, there is no decomposition $\mathbf{y} = \sum_{i=1}^{t} \mathbf{y}'_i$ such that all consumers in $\mathbf{y}$ have* ***strictly*** *higher utility in the segmentation $\{\mathbf{x}_1 - \mathbf{y}_1 + \mathbf{y}'_1, \cdots, \mathbf{x}_t - \mathbf{y}_t + \mathbf{y}'_t\}$ than in $\sigma(\mathbf{x}^*)$.*

In the definition, we require each group to have a positive measure, $\|\mathbf{y}\|_1 > 0$. This definition is quite general since we impose no restrictions on the composition of $\mathbf{y}$. The group can be small or large and can contain consumers with different valuations and from different markets. Following previous studies, a group of consumers will approve the deviation if all consumers obtain a higher utility.[19]

We assume the number of tags or markets is given, and any group of consumers cannot build a new market. This assumption is not a loss of generality since the consumer with the lowest valuation within the group must have zero utility by establishing a new market. Hence, building a new market can be considered, but consumers will find it meaningless.

The concept of stable segmentation can be easily extended to situations where the consumer can only delete his account instead of moving to other markets. Then, every welfare consequence achievable by Lemma 1 is attainable by stable segmentation.[20]

To facilitate our analyses, we introduce a relaxed concept called *weak-stable*, which is only robust to a subset of possible deviations. Here, we consider a small group of identical consumers from the same market. The measure of them, denoted by ε, is positive but arbitrarily close to zero.

Definition 2 (Weak-stable). *A segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$ is **weak-stable**, if for a **small** group of consumers **with the same valuation** in market $\mathbf{x}_i$, it is not profitable for them to deviate to any other market.*

By definition, every stable segmentation must be weak-stable. Then, the concept of weak-stable helps us provide bounds for the welfare consequences of stable segmentation.

3 Stable Market Segmentation

We now characterize possible welfare consequences of stable market segmentation, as in Section 3.1. As the key step, we show that pricing the uniform monopoly price must be optimal in all markets within any *weak-stable* segmentation. This property helps characterize the bounds of welfare consequences. Meanwhile, these bounds can be achieved by construction and are thus tight. Compared with the uniform monopoly, a stable segmentation can raise the consumer surplus without hurting the producer. Moreover, no

[19]See Barberà, Berga and Moreno (2016) for an example.

[20]Consider an arbitrary segmentation with all consumers logging in. The only option for consumers is account deletion, which automatically builds a new market, reveals the reservation price, and leads to a zero payoff. Hence, any group of consumers has no incentive to delete their account, implying stability.

individual consumer is worse off, indicating a Pareto improvement. Section 3.2 further provides necessary and sufficient conditions for a segmentation to be stable.

3.1 Welfare Consequences

In the first half of this subsection, we seek to find a necessary condition for weak-stable segmentation. Then this condition is also necessary for stable segmentation. Before the formal investigation, we make an observation, which plays a vital role in our analysis.

Observation 1. $\lim_{\varepsilon \to 0^+} \phi^{\min}(\mathbf{x} + \varepsilon \mathbf{e}_k)$ *is optimal in market* $\mathbf{x}$ *for any* k. *In particular,*

- *when* $v_k < \phi^{\min}(\mathbf{x})$, $\lim_{\varepsilon \to 0^+} \phi^{\min}(\mathbf{x} + \varepsilon \mathbf{e}_k) = \phi^{\min}(\mathbf{x})$;
- *when* $v_k \geq \phi^{\min}(\mathbf{x})$, $\lim_{\varepsilon \to 0^+} \phi^{\min}(\mathbf{x} + \varepsilon \mathbf{e}_k)$ *is the greatest optimal price in* $\mathbf{x}$ *that is no larger than* v_k.

Proof. The proof is relegated to Appendix A. □

Consider a small group of consumers entering the market $\mathbf{x}$. Observation 1 states that the resulting price in any market after an entry must be optimal before the entry. If pricing v is not optimal in market $\mathbf{x}$ before the entry, pricing v must be strictly worse than pricing $\phi^{\min}(\mathbf{x})$ by a positive size. Meanwhile, an entry can increase the revenue of pricing v, at most, infinitesimally. Hence, v cannot be optimal after the entry.

Among all possible stable segmentations, one simple market segmentation is the aggregate market itself (or called no segmentation), as shown in Remark 1.

Remark 1. $\sigma(\mathbf{x}^*) = \{\mathbf{x}^*\}$ *is stable.*

Remark 1 further implies that stable segmentation always exists.[21] If consumers are bounded rational such that they cannot forecast other markets' prices after any deviation, only segmentations with uniform prices can be stable.

However, rational consumers can recognize that prices may change after a deviation. Then, whether a stable segmentation exists with different prices in different markets? Suppose $\phi^{\min}(\mathbf{x}_i) < \phi^{\min}(\mathbf{x}_j)$. For the market segmentation to be stable, we need it to be weak-stable. Then, a small group of consumers with the same valuation from $\mathbf{x}_j$ has no incentive to enter $\mathbf{x}_i$. The price in the ith market after the entry must be at least $\phi^{\min}(\mathbf{x}_j)$ to rule out profitable deviations, implying that $\mathbf{x}_i$ should be somewhat *sensitive*.

[21] The segmentation is always weak-stable if prices are uniform. However, the segmentation may be unstable even if prices are uniform. See Example 2 later.

Straightforwardly, $\mathbf{x}_i$ should have multiple optimal prices that includes $\phi^{\min}(\mathbf{x}_j)$, as shown in Lemma 2. In fact, the price in $\mathbf{x}_i$ will be exactly $\phi^{\min}(\mathbf{x}_j)$ after such a deviation.

Lemma 2. *If segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$ is (weak-)stable and there exist two markets with different prices, $\phi^{\min}(\mathbf{x}_i) < \phi^{\min}(\mathbf{x}_j)$, then $\phi^{\min}(\mathbf{x}_j)$ is also optimal in market $\mathbf{x}_i$.*

Proof. The proof is relegated to Appendix A. □

Lemma 2 points out the possibility of going beyond the uniform monopoly outcome. The restrictions, however, are quite strict for a stable segmentation to have market-dependent prices. After eliminating unstable segmentations by Lemma 2, we rigorously demonstrate all possible welfare consequences for stable segmentations in Theorem 1.

Theorem 1. *The surplus of the producer and consumers (π, u) can be achieved by the stable segmentation if and only if $\pi = \pi^*$ and $u \in [u^*, \overline{w} - \pi^*]$. No consumer is worse off compared with the uniform monopoly outcome.*

We start with necessity. Consider a stable segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$. Lemma 3, the key step for necessity, states that the maximum price across all markets ($\max\left\{\phi^{\min}(\mathbf{x}_i)\right\}_{i=1}^{t}$) must equal to the uniform monopoly price v^*.

Lemma 3. *For any stable segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$, $\max\left\{\phi^{\min}(\mathbf{x}_i)\right\}_{i=1}^{t} = v^*$.*

Proof. The proof is relegated to Appendix A. □

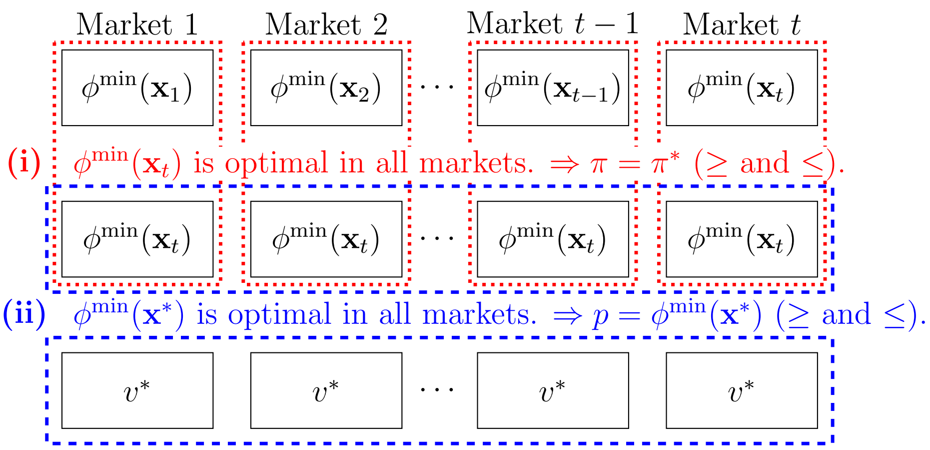

Figure 3: Why $\max\left\{\phi^{\min}(\mathbf{x}_i)\right\}_{i=1}^{t} = v^*$?

Figure 3 graphically illustrates the main logic to prove Lemma 3. The three panels represent three different pricing strategies. In the upper panel, the producer charges $\phi^{\min}(\mathbf{x}_i)$

in market i. Without loss of generality, we assume $\phi^{\min}(\mathbf{x}_1) \leq \phi^{\min}(\mathbf{x}_2) \leq \cdots \leq \phi^{\min}(\mathbf{x}_t)$. The producer surplus, in this case, is denoted by π. In the middle panel, the producer charges a uniform price $\phi^{\min}(\mathbf{x}_t)$ in all markets. Denote the producer surplus in this case by π'. In the lower panel, the producer charges the minimum optimal uniform price v^* in all markets and obtains the uniform monopoly profit π^*.

- We first argue that the three pricing strategies yield the same producer surplus. By Lemma 2, $\phi^{\min}(\mathbf{x}_t)$ should be optimal in every market. Thus, the pricing strategies in the upper and middle panels result in the same producer surplus, that is, $\pi = \pi'$. Evidently, $\pi' \leq \pi^*$, since the latter is the uniform monopoly profit and the former is the profit by charging a uniform price $\phi^{\min}(\mathbf{x}_t)$. Moreover, from Lemma 1, we know that $\pi^* \leq \pi$. Thus, we must have $\pi = \pi' = \pi^*$.
- Now, we argue that $\phi^{\min}(\mathbf{x}_t) = v^*$. On the one hand, since $\pi' = \pi^*$, $\phi^{\min}(\mathbf{x}_t)$ must be an optimal uniform price, suggesting that it should be greater than or equal to v^*. On the other hand, v^* should be optimal in every market within $\sigma(\mathbf{x}^*)$.[22] In particular, v^* should be optimal in market t. Hence, v^* cannot be strictly lower than $\phi^{\min}(\mathbf{x}_t)$.

Lemma 3 reveals one main insight: Circulating consumers enabled by data regulations prevent the producer from pricing higher than v^*. All results shown in Theorem 1 are direct implications of Lemma 3.

- **Producer surplus**. Since v^* is optimal in all markets within segmentation $\sigma(\mathbf{x}^*)$, the producer surplus is unchanged if charging v^* to all markets, resulting $\pi = \pi^*$.
- **Pareto improvement**. No consumer faces a higher price than v^*.
- **Consumer surplus**. By Pareto improvement, consumer surplus is at least the uniform monopoly outcome, $u \geq u^*$. Meanwhile $u \leq \overline{w} - \pi^*$ by Lemma 1.

For sufficiency, we construct stable segmentations to realize every (π^*, u) with $u \in [u^*, \overline{w} - \pi^*]$. For one extreme case, no segmentation can achieve $u = u^*$. The other extreme case, namely $u = \overline{w} - \pi^*$, is realized by greedy procedures we will explain immediately.

The constructive approach necessarily requires non-uniform pricing, indicating that multiple optimal prices may exist in some markets (Lemma 2). Since we seek to implement the extreme case with $u = \overline{w} - \pi^*$, we enable multiple optimal prices in an extreme way: *For every market within the segmentation, the producer is indifferent between charging*

[22]Suppose not. Then, there exists a market i such that charging v^* yields a strictly smaller profit than charging $\phi^{\min}(\mathbf{x}_t)$. Since $\pi' = \pi^*$, there must be a market j such that charging v^* yields a strictly larger profit than charging $\phi^{\min}(\mathbf{x}_t)$. This contradicts the fact that $\phi^{\min}(\mathbf{x}_t)$ is optimal in every market.

any price inside the support. This trenchant requirement coincides with the segmentation based on *extremal market* (Bergemann, Brooks and Morris, 2015).[23]

For a set of valuations $S \subseteq V$, the extremal market $\mathbf{x}^S(a)$ with total share being a, is defined as

$$x_i^S(a) = \begin{cases} 0 & v_i \notin S \\ a \min S \left(\frac{1}{v_i} - \frac{1}{\mu(v_i, S)} \right) & v_i \in S \end{cases}$$

where $\mu(v_i, S)$ denotes the smallest element in S higher than v_i, and $1/\mu(\max S, S) = 0$. We can verify that the producer is indifferent to charging any valuation that appears in an extremal market.[24] Hence, extremal markets are sensitive to entry and helpful for constructing stable segmentations.

An extremal market also *prevents the entry of a group of outside consumers who have a common value in the support*, as formally stated in Remark 2.

Remark 2. *Consider a group of consumers whose valuations all belong to S. Some of them would receive zero utility when joining $\mathbf{x}^S(a)$.*

With these good properties in mind, we generate markets iteratively. Write **supp** for the support of a distribution. First, pack as many consumers as possible into an extremal market $\mathbf{x}^{\mathbf{supp}\{\mathbf{x}^*\}}(a_1)$ until running out of consumers with some valuation. The residual market is $\mathbf{x}^{(1)} = \mathbf{x}^* - \mathbf{x}^{\mathbf{supp}\{\mathbf{x}^*\}}(a_1)$. Second, pack as many consumers as possible from $\mathbf{x}^{(1)}$ into an extremal market $\mathbf{x}^{\mathbf{supp}\{\mathbf{x}^{(1)}\}}(a_2)$. The residual market is thus $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} - \mathbf{x}^{\mathbf{supp}\{\mathbf{x}^{(1)}\}}(a_2)$. In each round k, we will obtain an extremal market $\mathbf{x}^{\mathbf{supp}\{\mathbf{x}^{(k-1)}\}}(a_k)$ and a residual market $\mathbf{x}^{(k)}$. This iteration is terminated when the remaining market becomes $\mathbf{0}$. Since $|\mathbf{supp}\{\mathbf{x}^{(i+1)}\}| < |\mathbf{supp}\{\mathbf{x}^{(i)}\}|$, there is at most K iterations, and the resulting segmentation is called *greedy segmentation* $\sigma^{\text{Greedy}}(\mathbf{x}^*) = \{\mathbf{x}_1^{\text{Greedy}}, \cdots, \mathbf{x}_t^{\text{Greedy}}\}$. The greedy segmentation reaches $u = \overline{w} - \pi^*$, and more importantly, is stable.

Remark 3. *The greedy segmentation $\sigma^{\text{Greedy}}(\mathbf{x}^*)$ is stable.*

For sufficiency of Theorem 1, it remains to construct a family of stable segmentations that can achieve any intermediate $u \in (u^*, \overline{w} - \pi^*)$ based on greedy segmentation $\sigma^{\text{Greedy}}(\mathbf{x}^*)$. Given $\alpha \in [0, 1]$ and $k \in \{1, \cdots, t-1\}$, we merge those markets that are

[23] Bergemann, Brooks and Morris (2015) construct two segmentations to implement $u = \overline{w} - \pi^*$: a segmentation based on direct market and a segmentation based on extremal market. In this paper, we formally prove the stability of the latter one. However, the former generates an unstable segmentation.

[24] Charging any price in S will result in a profit of $a \min S$.

generated later in greedy procedures to obtain a new segmentation, denoted by $\sigma^{\alpha,k}(\mathbf{x}^*)$:

$$\left\{ \underbrace{\mathbf{x}_1^{\text{Greedy}}, \cdots, \mathbf{x}_{k-1}^{\text{Greedy}}, \alpha \mathbf{x}_k^{\text{Greedy}}}_{\text{Extremal Markets}}, \underbrace{\sum_{j=k+1}^{t} \mathbf{x}_j^{\text{Greedy}} + (1-\alpha)\mathbf{x}_k^{\text{Greedy}}}_{\text{Last Market}} \right\}.$$

Notably, the first k markets are extremal. Meanwhile, the minimum optimal price of the last market is v^*, since (i) v^* is optimal in every market generated by the greedy procedures, and (ii) v^* is the minimum optimal price in $\mathbf{x}_t^{\text{Greedy}}$ (Lemma 3).[25]

Remark 4. *The constructed segmentation $\sigma^{\alpha,k}(\mathbf{x}^*)$ is stable.*

To show that the constructed segmentation is stable, we iteratively consider the market $\mathbf{x}_i^{\text{Greedy}}$ from $i = 1$ to $i = k$. In the ith iteration, we find that no consumer wants to (i) deviate *from* $\mathbf{x}_i$ *to* markets with higher indices, since the price in $\mathbf{x}_i$ is the minimum valuation among later generated markets; (ii) deviate *to* $\mathbf{x}_i$ *from* markets with higher indices, since all consumers in later markets have their valuations on the support of $\mathbf{x}_i$ (Remark 2). Figure 4 graphically illustrates all k rounds of iterations.

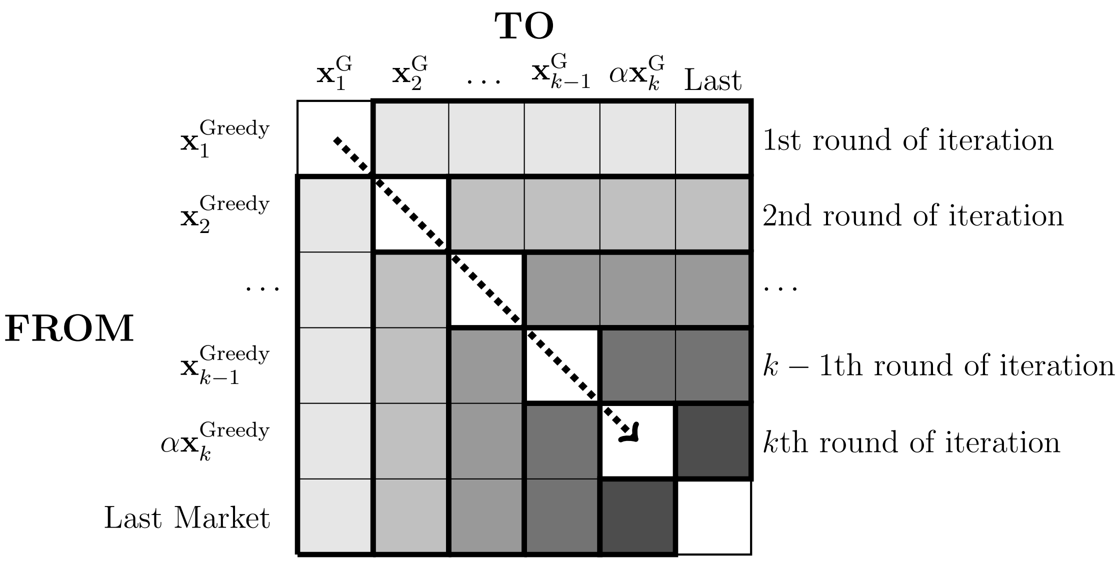

Figure 4: Iteration Process of Remark 4

Finally, we show that any intermediate $u \in (u^*, \overline{w} - \pi^*)$ can be achieved. Let Ψ denote the share of the last market, $\Psi(\sigma^{\alpha,k}(\mathbf{x}^*)) = \left\| \sum_{j=k+1}^{t} \mathbf{x}_j^{\text{Greedy}} + (1-\alpha)\mathbf{x}_k^{\text{Greedy}} \right\|_1$. Given any

[25] To see why the price of the last market is v^*, it suffices to show that (I) pricing any $v < v^*$ is strictly worse than v^* and (II) pricing any $v > v^*$ is not better than v^*. (I) holds since v^* is optimal in all markets and any $v < v^*$ is strictly worse than v^* in $\mathbf{x}_t^{\text{Greedy}}$ since $v^* = \phi^{\min}(\mathbf{x}_t^{\text{Greedy}})$. (II) holds immediately from (i).

$\psi \in \left[\|\mathbf{x}_t^{\text{Greedy}}\|_1, 1\right]$, there exists a unique $\sigma^{\alpha,k}(\mathbf{x}^*)$ such that $\Psi(\sigma^{\alpha,k}(\mathbf{x}^*)) = \psi$. Meanwhile, the consumer surplus is continuous (and monotonic) in ψ.[26] Since the consumer surplus is $\overline{w} - \pi^*$ when $\psi = \|\mathbf{x}_t^{\text{Greedy}}\|_1$ and u^* when $\psi = 1$, all consumer surplus within $[u^*, \overline{w} - \pi^*]$ can be achieved. This ends the proof of Theorem 1.

So far, we have gained the first insight into the structure of stable segmentations. All constructed stable segmentations consist of a market with a uniform monopoly price, namely the *last market*, and some discount markets. Defining the last market as the anonymous market, we can interpret the above market structure intuitively in a loose way. The anonymous market includes those consumers who exercise the right to be entirely forgotten. These consumers refuse to provide any information, thus facing the uniform monopoly price. Meanwhile, consumers in each discount market provide some information to get preferential treatment.

For a more rigorous exposition of our above interpretation, consider the following refinement rule on stable segmentations about the information-price tradeoff. For each segmentation, exactly one of the markets should be marked as the *anonymous market*, containing those consumers who delete their accounts. Besides the condition mentioned in Definition 1, we introduce the following *no-logout* condition, featuring the incentive to delete the account: If a group of consumers can purchase the good at a weakly lower price without login, they will do so to prevent information leakage.[27]

Definition 3 (No-logout Condition)**.** *Assume one market within the segmentation represents the* ***anonymous market****, denoted by* $\mathbf{x}_t$*. Consumers with a successful trade outside* $\mathbf{x}_t$ *cannot deviate to* $\mathbf{x}_t$ *to face* ***a weakly lower price****.*

After imposing the no-logout condition as a refinement criterion, all surviving stable segmentations include exactly one market with price v^*, namely the anonymous market. In fact, for any stable segmentation without refinement, merging all markets with price v^* and calling it the anonymous market will result in a stable segmentation satisfying the no-logout condition. Therefore, welfare consequences remain unchanged before and after the refinement. The no-logout condition is introduced merely for better interpretation.

[26] As ψ increases by $\mathrm{d}\psi$, those $\mathrm{d}\psi$ consumers, once in the second last market $\alpha\mathbf{x}_k^{\text{Greedy}}$, face price v^* instead, while other consumers face the same price as before. Hence, the decline of consumer surplus is solely influenced by those changed consumers. Since changed consumers are distributed proportional to $\mathbf{x}_k^{\text{Greedy}}$, the loss of consumer surplus is bounded by $\mathrm{d}\psi\left(v^* - \phi^{\min}(\mathbf{x}_k^{\text{Greedy}})\right)$: Continuity holds.

[27] We do not replicate this condition on consumers without a purchase for two reasons. On the one hand, information leakage primarily occurs during a transaction. On the other hand, only the result with no segmentation survives if we impose restrictions on consumers with no trade.

For illustration purposes, we use the following example to explain our results on welfare consequences (Theorem 1).

Example 1. *There are three valuations $V = \{1, 2, 3\}$, with equal proportions. Thus, $K = 3$, $v_k = k$, and $\mathbf{x}^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It is easy to obtain that $\overline{w} = 2, \pi^* = \frac{4}{3}$ and $u^* = \frac{1}{3}$.*

If the producer wants to impose first-degree price discrimination, the aggregate market can be split as $\mathbf{x}_1 = (\frac{1}{3}, 0, 0), \mathbf{x}_2 = (0, \frac{1}{3}, 0), \mathbf{x}_3 = (0, 0, \frac{1}{3})$. Here, social welfare is maximized and fully extracted by the producer. However, if consumers can change their tags and reallocate themselves into different markets, this segmentation is no longer stable.

The greedy segmentation is shown in Table 1. Since all consumers purchase the product, social welfare is maximized. Meanwhile, the producer surplus equals π^*, implying that the consumer surplus reaches the buyer-optimal level $u = \overline{w} - \pi^*$. Hence, this example thus numerically illustrates how data regulations transfer surplus from the producer to consumers without hurting social welfare.

If we want to realize $u = \frac{1}{2} \in [\frac{1}{3}, \frac{2}{3}] = [u^*, \overline{w} - \pi^*]$, the segmentation is constructed as $\left\{\frac{1}{2}\mathbf{x}_1^{\text{Greedy}}, \frac{1}{2}\mathbf{x}_1^{\text{Greedy}} + \mathbf{x}_2^{\text{Greedy}} + \mathbf{x}_3^{\text{Greedy}}\right\}$, where $k = 1, \alpha = \frac{1}{2}$ and $\Psi = \frac{2}{3}$.

	$v_1 = 1$	$v_2 = 2$	$v_3 = 3$	optimal price(s)	$\phi^{\min}(\cdot)$	profit
$\mathbf{x}_1^{\text{Greedy}}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{2}{9}$	1, 2, and 3	1	$\frac{2}{3}$
$\mathbf{x}_2^{\text{Greedy}}$	0	$\frac{1}{18}$	$\frac{1}{9}$	2 and 3	2	$\frac{1}{3}$
$\mathbf{x}_3^{\text{Greedy}}$	0	$\frac{1}{6}$	0	2	2	$\frac{1}{3}$
$\mathbf{x}^*$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	2	2	$\frac{4}{3}$

Table 1: Greedy segmentation of the example market

3.2 Necessary and Sufficient Condition

In the following, we aim to provide necessary and sufficient conditions regarding the structure of stable segmentations. To begin with, we consider weak-stable segmentations.

3.2.1 Weak-Stable

Recall that Lemma 2 provides a necessary condition for weak-stable segmentation; however, it is not sufficient. For example, consider an aggregate market $\mathbf{x}^* = (\frac{1}{4}, \frac{1}{8}, \frac{5}{8})$ with the valuation set $V = \{1, 2, 3\}$. The segmentation $\{\mathbf{x}_1, \mathbf{x}_2\}$ with $\mathbf{x}_1 = (\frac{1}{4}, 0, \frac{1}{8})$ and $\mathbf{x}_2 = (0, \frac{1}{8}, \frac{1}{2})$ satisfies Lemma 2. However, the consumer with a valuation of 2 in $\mathbf{x}_2$ still has the incentive to deviate to $\mathbf{x}_1$.

The following proposition provides the necessary and sufficient condition of weak-stable market segmentation, which can also serve as a tractable verification condition for weak-stable segmentation. If Proposition 1 fails, the segmentation is not (weak-)stable.

Proposition 1. *The segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$ is weak-stable if and only if the following **indifference** condition holds: if there exist two markets with different prices, $\phi^{\min}(\mathbf{x}_i) < \phi^{\min}(\mathbf{x}_j)$, then for all $v_k \in \mathbf{supp}\{\mathbf{x}_j\} \cap \left(\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_j)\right]$, v_k is also optimal in market $\mathbf{x}_i$.*

Proof. The proof is relegated to Appendix A. □

Notice that the necessary condition mentioned in Lemma 2 only requires $\phi^{\min}(\mathbf{x}_j)$ to be optimal in market $\mathbf{x}_i$. In contrast, to make it sufficient, any valuation on the support of $\mathbf{x}_j$ and interval $\left(\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_j)\right]$ should also be optimal.

Proposition 1 manifests that a weak-stable segmentation occurs if and only if all consumers with zero utility have no incentives to deviate. The intuition is straightforward as follows. The necessity part shares a similar logic with Lemma 2: Any movement from a market $\mathbf{x}_j$ with a higher price to another market $\mathbf{x}_i$ with a lower price will immediately raise the price of $\mathbf{x}_i$, provided that the segmentation is weak-stable. For sufficiency, Proposition 1 can be understood from the consumer's perspective. Observation 1 implies that the price cannot decrease after an entry. Hence, any consumer merely ignores all markets with a weakly higher price than (i) the price in his current market or (ii) his valuation. The consumer with value v_k in $\mathbf{x}_j$ only searches for market $\mathbf{x}_i$ such that $\phi^{\min}(\mathbf{x}_i) < \phi^{\min}(\mathbf{x}_j)$ and $v_k > \phi^{\min}(\mathbf{x}_i)$. However, $\min\left(v_k, \phi^{\min}(\mathbf{x}_j)\right)$ is also optimal in all those markets that meet the above criteria, suggesting a non-profitable deviation.

3.2.2 Stable

The set of stable segmentation is a proper subset of weak-stable segmentation. In the following, we provide an example to illustrate that a weak-stable segmentation may not be stable.

Example 2. *There are three valuations $V = \{1, 2, 3\}$, with aggregate market $\mathbf{x}^* = (\frac{6}{11}, \frac{1}{11}, \frac{4}{11})$. The segmentation listed in Table 2 is weak-stable but not stable.*

The above segmentation satisfies the indifference condition shown in Proposition 1, it should be weak-stable. However, it is not stable. All consumers with value 2 in the market

	$v_1 = 1$	$v_2 = 2$	$v_3 = 3$	$\phi^{\min}(\cdot)$	profit
$\mathbf{x}_1$	$\frac{3}{11}$	$\frac{1}{22}$	$\frac{2}{11}$	3	$\frac{6}{11}$
$\mathbf{x}_2$	$\frac{3}{11}$	$\frac{1}{22}$	$\frac{2}{11}$	3	$\frac{6}{11}$
$\mathbf{x}^*$	$\frac{6}{11}$	$\frac{1}{11}$	$\frac{4}{11}$	3	$\frac{12}{11}$

Table 2: A weak-stable segmentation but not stable

$\mathbf{x}_2$ will be better off after moving to $\mathbf{x}_1$. The associated price in market $\mathbf{x}_1$ will decrease to 1, implying that all those consumers are strictly better off.

The above example demonstrates that the set of stable segmentation can be strictly smaller than weak-stable segmentation. Consequently, the verification condition for stable segmentation is totally different from weak-stable segmentation.

Proposition 2. *The segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$ is stable if and only if the following* ***no-inflow*** *condition holds: for any market $\mathbf{x}_i \in \sigma(\mathbf{x}^*)$, there is no group of consumers $\mathbf{y} \neq \mathbf{0}$ from other markets such that all consumers in $\mathbf{y}$ have a strictly higher utility in market $\mathbf{x}_i + \mathbf{y}$.*

Proof. The proof is relegated to Appendix A. □

Unlike weak-stable segmentation, stable segmentation is even impossible to verify directly through definition, because there are infinite possible groups $\mathbf{y}$. Nevertheless, since Proposition 2 only specifies a certain class of deviation, a stable segmentation must be immune to them. Hence, we need only to specify the sufficiency side of Proposition 2. Sufficiency comes from the negative statement: For any segmentation that is not stable, we can select a market and find a group of consumers $\mathbf{y}$ from other markets such that all consumers in $\mathbf{y}$ have a strictly higher utility.

We can construct a protocol to verify the group stability based on Proposition 2.

1. Select a market $\mathbf{x} \in \sigma(\mathbf{x}^*)$.
2. Select a target price v_k to be reached after deviation.
3. Consumers who can benefit from a deviation to market $\mathbf{x}$ must satisfy that (i) valuation is larger than v_k and (ii) price of the initial market is larger than v_k. Gather those consumers in $\mathbf{z}$.
4. Verify whether there exists a group of consumers $\mathbf{y}$ from $\mathbf{z}$ such that the price after the entry is v_k. Mathematically, test whether a group of consumers $\mathbf{y} \leq \mathbf{z}$ such that $\phi^{\min}(\mathbf{x} + \mathbf{y}) = v_k$.
5. Repeat Steps 2 to 4 for all possible $v_k \in V$.

6. Repeat Steps 2 to 5 for all markets in $\sigma(\mathbf{x}^*)$.

The above protocol presents an efficient algorithm to certify whether a given segmentation is stable by checking no-inflow conditions for all markets. This can be reduced to a linear programming problem.[28]

4 Social-Optimal Stable Segmentation

A segmentation is referred to as *buyer-optimal* if the consumer surplus equals $\overline{w} - \pi^*$, which must be *social-optimal* as well.[29] Thus, Theorem 1 reveals that the buyer-optimal outcome can survive under our framework.

Remark 5. *There exists a stable segmentation that implements* $(\pi, u) = (\pi^*, \overline{w} - \pi^*)$.

This observation reveals the positive side of price discrimination in the presence of data regulations: Allowing price discrimination protects consumers rather than hurts consumers. Without price discrimination, the producer will adopt the uniform monopoly price, causing a deadweight loss (equaling to $\overline{w} - \pi^* - u^*$) that is entirely borne by consumers.

In this section, we will characterize those stable and social-optimal segmentations thoroughly. In particular, Section 4.1 shows that weak-stable and stable are identical concepts for social-optimal segmentations. Section 4.2 presents the common structure of stable, social-optimal, and direct segmentations, which paves the way for deriving all such segmentations in Section 4.3.

4.1 Equivalence between Weak-Stable and Stable

If we restrict our attention to social-optimal segmentations, the only attainable outcome through stable segmentation is the buyer-optimal outcome. Any social-optimal outcomes other than the buyer-optimal outcome must not be stable.

Moreover, not all market segmentations that achieve the buyer-optimal outcome are stable. Consider two algorithms in Bergemann, Brooks and Morris (2015) that implement the buyer-optimal outcome. One algorithm generates an extremal segmentation, which is proved stable by Remark 3. The other algorithm generates an unstable direct segmentation since the conditions in Proposition 1 are not satisfied in that segmentation.

[28] In social choice theory, it is intractable to find group manipulation in most social choice functions.

[29] See Bergemann, Brooks and Morris (2015); Roesler and Szentes (2017); Deb and Roesler (2021) for works on buyer-optimal outcomes.

Based on the above considerations, it is necessary to clarify the condition of social-optimal segmentation to be stable. Although conditions in Proposition 1 and Proposition 2 are still valid, we can find a simpler necessary and sufficient condition. In particular, we find that *weak-stable segmentation* and *stable segmentation* coincide with each other among those social-optimal outcomes, and that the necessary condition mentioned in Lemma 2 is now sufficient, as summarized in Proposition 3.

Proposition 3. *Consider a social-optimal segmentation* $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$*. The following three statements are equivalent:*

(1) $\sigma(\mathbf{x}^*)$ *is weak-stable.*

(2) $\sigma(\mathbf{x}^*)$ *is stable.*

(3) $\sigma(\mathbf{x}^*)$ *satisfies:* $\underline{\phi^{\min}(\mathbf{x}_i) < \phi^{\min}(\mathbf{x}_j)}$ *implies* $\underline{\phi^{\min}(\mathbf{x}_j) \text{ is optimal in market } \mathbf{x}_i}$.

Proof. Since (1) ⇒ (3) is valid by Lemma 2, and (2) ⇒ (1) holds trivially, we need only to prove (3) ⇒ (2). The proof is relegated to Appendix A. □

4.2 Markets within a Direct Segmentation

4.2.1 Direct Segmentation

Proposition 4 presents the stable-preserving property of merging markets with the same price. *Merging markets* refers to the procedure of replacing $\mathbf{x}_i$ and $\mathbf{x}_j$ in the original segmentation by $\mathbf{x}_i + \mathbf{x}_j$. However, it remains unknown whether the stable-preserving property holds for non-social-optimal stable segmentation. Our conjecture is positive.

Proposition 4. *If segmentation* $\sigma(\mathbf{x}^*)$ *is stable and social-optimal, and there exists* $\mathbf{x}_i, \mathbf{x}_j$ *such that* $\phi^{\min}(\mathbf{x}_i) = \phi^{\min}(\mathbf{x}_j)$*, then* $\sigma'(\mathbf{x}^*) = \sigma(\mathbf{x}^*) \backslash \{\mathbf{x}_i, \mathbf{x}_j\} \cup \{\mathbf{x}_i + \mathbf{x}_j\}$ *is also stable.*

Proof. The proof is relegated to Appendix A. □

Proposition 4 can simplify every segmentation to a *direct segmentation.*

Definition 4 (Direct Segmentation). *A segmentation* $\sigma(\mathbf{x}^*)$ *is* ***direct****, if* $|\sigma(\mathbf{x}^*)| \leq K$ *and* $\phi^{\min}(\mathbf{x}) \neq \phi^{\min}(\mathbf{x}')$ *for any* $\mathbf{x}, \mathbf{x}' \in \sigma(\mathbf{x}^*)$.

Since the implementation cost of direct segmentation is lower due to fewer markets, it is more convenient for the market designer to realize stable, social-optimal, and direct (SSD for short) segmentation.

4.2.2 Markets within an SSD Market

Proposition 3 clarifies that any weak-stable segmentation is also stable once it is social-optimal. However, all the above analyses focus on aggregate-level welfare consequences but not market-specific features. It is unclear whether the various SSD segmentations have different welfare distributions across all markets. For clarity, we concentrate on direct segmentations.

For an SSD segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$, we define the revenue profile over markets as $\{\pi_1, \cdots, \pi_t\}$ where π_i is the producer's profit in market $\mathbf{x}_i$, and the price profile over markets as $P(\mathbf{x}^*) = \{\phi^{\min}(\mathbf{x}_1), \cdots, \phi^{\min}(\mathbf{x}_t)\}$. The following proposition provides a strong result on welfare distributions across markets.

Proposition 5. *The price profile $\{\phi^{\min}(\mathbf{x}_1), \cdots, \phi^{\min}(\mathbf{x}_t)\}$ and revenue profile $\{\pi_1, \cdots, \pi_t\}$ are identical for all stable, social-optimal, and direct (SSD) segmentations.*

Proof. The proof is relegated to Appendix A. □

Realizing that all SSD segmentations share the same price profile and revenue profile, we can compute them straightforwardly by running the greedy segmentation procedure mentioned in Section 3.1. By merging all markets with the same price, we can obtain an SSD segmentation and pin down the price profile and the revenue profile.

The characterization of the price profile and the revenue profile can also be illustrated geometrically. Define $\hat{\pi}(v_i) = v_i \sum_{j=i}^{K} x_j^*$ as the *revenue function* of charging $v_i \in V$ uniformly in the aggregate market. Apparently, $\pi^* = \max_{v_i} \hat{\pi}(v_i)$.

Example 3. *There are five possible valuations $V = \{1, 2, 3, 4, 5\}$, with probability distribution $\mathbf{x}^* = (0.1, 0.3, 0.1, 0.3, 0.2)$. Thus, $\hat{\pi}(1) = 1, \hat{\pi}(2) = 1.8, \hat{\pi}(3) = 1.8, \hat{\pi}(4) = 2$ and $\hat{\pi}(5) = 1$. By definition $v^* = 4$ and $\pi^* = 2$.*

Figure 5 plots the revenue function of $\mathbf{x}^* = (0.1, 0.3, 0.1, 0.3, 0.2)$ with $V = \{1, 2, 3, 4, 5\}$ and its *left concave closure*. The concave closure of $\hat{\pi}$, defined in $[v_1, v_K]$, is the smallest concave function that is everywhere weakly greater than $\hat{\pi}$ (Kamenica and Gentzkow, 2011).[30] The left concave closure, defined in $[v_1, v^*]$ instead, is a monotonically increasing segmented linear function with several *inflection points* where the slope is changed. For example, the left concave closure shown by the red curve in Figure 5 has one inflection point v_2. The price profile thus contains v_1, v^* and the inflection point v_2 in Example 3.

[30] Notice that the revenue function is defined on a discrete set V, while the concave closure is defined in a closed interval $[v_1, v_K]$.

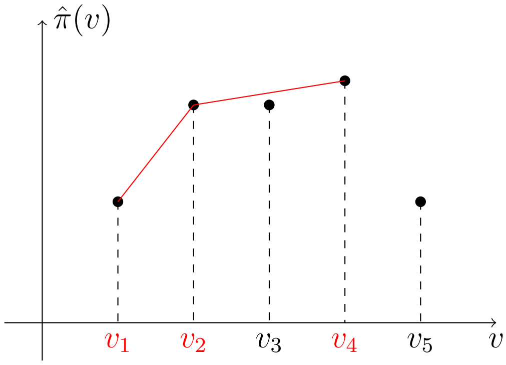

Figure 5: Price Profile

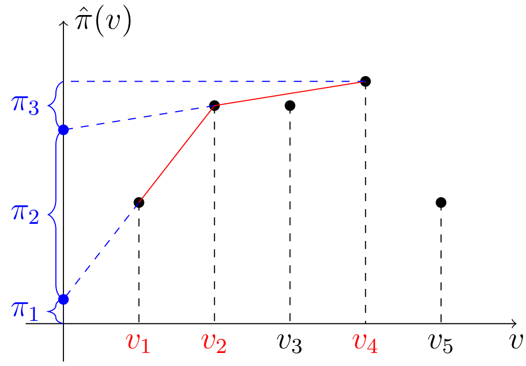

Figure 6: Revenue Profile

Figure 6 illustrates how to find the revenue profile based on Figure 5. Notice that the left concave closure is a segmented linear function. We can draw the intersection points for each line segment and the vertical axis, as shown by two blue points in Figure 5. Hence, we obtain a set of *intercept points*, including zero, π^*, and every intersection point. The distance between adjacent intercept points represents the revenue of one market. Appendix A will explain step-by-step how the price profile shown in Figure 5 and the revenue profile shown in Figure 5 are generated by the greedy procedures mentioned in Section 3.1.

4.3 Deriving All Direct Segmentations

Using the optimal price profile, we can derive all SSD market segmentations. Suppose the optimal price profile is $\{p_1, \cdots, p_t\} \subseteq \{v_1, \cdots, v^*\}$ with $v_1 = p_1 < \cdots < p_t = v^*$. We define the *virtual* segmentation for any segmentation such that consumers within $[p_i, p_{i+1})$ are treated as p_i. For Example 3, any virtual segmentation is a decomposition of the virtual aggregate market $\mathbf{x}^*_{\text{virtual}} = (0.1, 0.4, 0.5)$ with $V_{\text{virtual}} = \{1, 2, 4\}$. SSD segmentation exists uniquely in the virtual aggregate market, denoted by $\hat{\sigma}(\mathbf{x}^*_{\text{virtual}})$ which is the one generated by the greedy procedure mentioned in Section 3.1. Meanwhile, any SSD segmentation can be mapped into an identical virtual segmentation. We recover from $\hat{\sigma}(\mathbf{x}^*_{\text{virtual}})$ to obtain all possible SSD segmentations of $\mathbf{x}^*$. The recovery process is merely the assignment of consumers with a valuation in $V \backslash P(\mathbf{x}^*)$. The following two constraints should be guaranteed in the recovery process.

- **Balanced Distribution**. Consumers with $V \backslash P(\mathbf{x}^*)$ should be distributed in a balanced way such that pricing any valuation in $V \backslash P(\mathbf{x}^*)$ in each market cannot exceed its profit in the virtual segmentation.

- **Regularity**. All consumers with $V \backslash P(\mathbf{x}^*)$ are assigned to a specific market.

Finally, we use the aggregate market in Example 1 to illustrate our results for better understanding. First, the optimal price profile can be directly obtained using the greedy segmentation mentioned in Section 3.1. Thus, all SSD direct segmentations should include two markets with the price profile $P(\mathbf{x}^*) = \{1, 2\}$ and the revenue profile $\{\frac{2}{3}, \frac{2}{3}\}$. To derive all SSD segmentations, it remains to figure out the distribution of consumers with valuation $V \backslash P(\mathbf{x}^*) = \{3\}$. Let a denote the mass of consumers assigned to the market with price 1. Then, $\frac{1}{3} - a$ consumers go to the other market by regularity. Furthermore, pricing 3 should be never optimal in these two markets, namely

$$3a \leq \frac{2}{3}, \quad 3(\frac{1}{3} - a) \leq \frac{2}{3}.$$

Therefore, *all* SSD segmentations have the structure shown in Table 3, where $a \in [\frac{1}{9}, \frac{2}{9}]$.

	$v_1 = 1$	$v_2 = 2$	$v_3 = 3$	$\{v_2, v_3\}$	$\phi^{\min}$	profit
$\mathbf{x}_1$	$\frac{1}{3}$	$\frac{1}{3} - a$	a	$\frac{1}{3}$	1	$\frac{2}{3}$
$\mathbf{x}_2$	0	a	$\frac{1}{3} - a$	$\frac{1}{3}$	2	$\frac{2}{3}$
$\mathbf{x}^*$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	2	$\frac{4}{3}$

Table 3: All SSD segmentations of the market in Example 1

5 Discussions

5.1 Relaxing the Minimum Pricing Rule $\phi^{\min}$

Now we examine the robustness of our results. In this subsection, we relax the assumption of the minimum optimal pricing rule. The other two implicit assumptions, full information about the value distribution and full rationality, are discussed in Section 5.4.

To show the robustness of our results, we consider other rational tie-breaking rules instead of the minimum optimal pricing rule.

Definition 5 (Rational Pricing)**.** *A pricing strategy ϕ^O is **ex-post rational** if $\phi^O(\mathbf{x})$ is optimal in market $\mathbf{x}$.*

As an extreme case, we first consider the maximum optimal pricing strategy $\phi^{\max}$, which minimizes consumer surplus for tie-breaking.

Lemma 4. *If the producer adopts $\phi^{\max}$, the price of any market in any stable segmentation $\sigma(\mathbf{x}^*)$ is equal to the maximal optimal uniform monopoly price, $\phi^{\max}(\mathbf{x}^*)$.*

Proof. The proof is relegated to Appendix A. □

The surplus pair (π, u) under stable segmentation must be equal to the surplus of charging $\phi^{\max}(\mathbf{x}^*)$ uniformly. Consumer surplus is denoted by $u^\dagger$, which does not necessarily equal to u^* because $\phi^{\max}(\mathbf{x}^*)$ may not equal $\phi^{\min}(\mathbf{x}^*)$.

Proposition 6. *If the producer adopts an ex-post rational pricing rule ϕ^O, under any stable segmentation, the producer surplus is π^*, and the consumer surplus $u \geq u^\dagger$.*

Proof. The proof is relegated to Appendix A. □

Therefore, we fully recognize that any ex-post rational pricing rule (i) cannot change producer surplus, and (ii) cannot decrease consumer surplus compared with the (worst) uniform monopoly outcome. Moreover, allowing price discrimination is Pareto-improving compared with the (worst) outcome where price discrimination is prohibited.

5.2 Individual Deviation

Our results can be extended to the scenario where consumers behave independently without grouping. Although the simple continuum model in this paper is standard in the literature, consumers are finite in reality; thus, every individual consumer has a small but non-negligible market share. This kind of formalization already exists in the literature. For example, Cong and He (2019), who study the formalization of decentralized consensus in a blockchain system, assume that the number of agents is sufficiently large; however, each agent always has a small but positive share.

Having justified the positive measure of an individual consumer, we can treat each consumer alone as a particular form of group. Therefore, our main results can accommodate individual deviation, and the concept of weak-stable is reduced to the Nash equilibrium solution. That is to say, those decentralized consumers are not necessarily to be grouped to fight against the monopolist. A completely decentralized tag-editing is enough.

The remaining issue is that consumers need to be aware and believe they have positive market power, and their behavior can indeed change the market outcome. This issue will be addressed in Section 5.4 later.

5.3 Policy Implications

According to our analysis, stable market segmentation can achieve social-optimal and buyer-optimal outcomes, which is desirable for policymakers. However, some countries have forbidden tag-based price discrimination due to market power regulation and consumer privacy concerns. Big-data-enabled price discrimination is also in the regulatory crosshairs in China and remains controversial.[31] For example, using biased tags was once prohibited in the draft but was finally relaxed.

Our analysis suggests an alternative approach to protect consumers: Allowing consumers to strategically select their tags, which aligns with the spirit of the *General Data Protection Regulation*. Although consumers are still restricted from freely editing their tags in reality, our analyses suggest that giving consumers more freedom would improve overall efficiency. Therefore, we should stick to this direction. For example, merely deleting accounts on the app, corresponding to the right to be *entirely* forgotten, falls far short of protecting consumers. Instead, granting them the right to be *partially* forgotten is crucial to effective protection.[32]

For regulators, permitting tag editing is a Pareto improvement compared to allowing second-hand transactions, although both tools aim to prevent monopolists from exploiting consumers. The presence of a costless second-hand market reduces the market equilibrium to a uniform monopoly circumstance. In contrast, enabling circulation among markets, which is facilitated by data regulations, can further favor consumers without adversely affecting the monopolist.

Last but not least, promoting data trust, data brokers, or data mediators could play a significant role in solving the equilibrium selection problem among stable market segmentations, which can be taken into account by the policymaker. A data broker or mediator, whether motivated by self-interest or social (ESG) concerns, can help choose a social-optimal stable market segmentation, which is exactly buyer-optimal.

[31] The controversy can be reflected by the process of establishing regulations. The State Administration for Market Regulation in China seeks public comments on the draft rules that fine price discrimination in July 2021. However, the final version is still veiled. Note that the data regulations in China, which are already approved, seek public opinion in September 2021.

[32] If not, the producer can segment the market to implement first-degree price discrimination, where tags perfectly align with valuations. With all consumers labeled, no one wants to cancel the account voluntarily, as doing so reveals their willingness-to-pay and results in zero utility. Hence, the undesirable first-degree price discrimination outcome can still occur under stable segmentation.

5.4 Implicit Assumptions Revisited

Our analyses are conducted based on two implicit assumptions, *full information* (on value distributions but not on true valuations for all consumers) and *full rationality*, for both the producer and consumers.[33] The first assumption requires the producer to have accurate knowledge of consumers' value distribution before setting the price, while the second assumption assumes that the producer can respond to shifts in market conditions and adjust prices quickly. Examples such as Big-data-based price discrimination (including identity-dependent issuance and distribution of shopping coupons), dynamic pricing, and probabilistic selling, already ubiquitous in online environments, serve to demonstrate the seller's acumen in learning consumers' distribution and real-time market conditions and adjusting prices accordingly.[34] Meanwhile, growing evidence shows the prevalence of algorithmic pricing, whose automatic nature resolves our concerns.

For the demand side, we assume that consumers must be able to form rational beliefs about the actions taken by other consumers. In particular, each consumer can learn each market's value distribution and react to changes in the distribution. As a result, one may argue that stable market segmentation can hardly be implemented by the producer and consumers themselves due to bounded rationality or limited information. However, existing literature indicates that it is acceptable to assume that consumers can learn about prices in different markets. For example, Chen, Choe and Matsushima (2020) assume that active consumers can effortlessly observe personalized or uniform prices set by a producer and switch to another market by identity management.

The strong assumptions on the demand side are unavoidable outgrowths of incorporating a large number of strategic consumers, which moves one step further than previous analyses with a single consumer (see Roesler and Szentes (2017); Condorelli and Szentes (2020); Ali, Lewis and Vasserman (2023); Armstrong and Zhou (2022)). The problem becomes intractable once incomplete information is introduced to the classical framework. Therefore, we believe that our formalization, which at least serves as a theoretical benchmark for the real-world problem, is acceptable. Furthermore, mandated algorithmic trans-

[33] We do not need to assume that the seller can extract the true valuation for all consumers. Even if the seller can always learn the value distributions of consumers, she cannot infer their true valuations. Suppose a small group of consumers deviate to other markets. When comparing two distributions, the seller has forgotten consumers' positions in the previous segmentation. In other words, consumers of the same type can be rearranged across different markets, and the seller cannot detect it.

[34] See Airbnb (`https://padlifter.com/free-tips-and-resources/pricing/airbnb-smart-pricing-and-price-tips/`) and Delta Airlines (`https://business.time.com/2012/05/21/delta-overcharged-frequent-flyers-for-weeks-was-that-legal/`) for examples.

parency enforced by regulators (e.g., in the UK) is conducive to consumers' rationality, which also helps consumers recognize that they have positive market powers.[35]

While full information and full rationality may be idealized, data-related services can help address these concerns. Internet platforms that aggregate data from both the producer and consumers can facilitate the disclosure of information, making our assumption of complete information more reasonable. Meanwhile, these platforms typically possess enormous computing power, which can help ensure rational decision-making. As a result, the producer and consumers can rely on platform suggestions, which are constrained by incentive compatibility, to guide their actions. Think over WeChat, which connects sellers and buyers through mini-programs. WeChat, which operates benevolently to favor both sellers and customers, can gather information and correctly disseminate it.[36] Large PC/-phone manufacturers (e.g., Apple and Samsung) and giant data brokers (e.g., Acxiom and Bloomberg) can also play the above role. Furthermore, interactions between the producer and consumers are likely to occur repeatedly, facilitating learning and converging to the equilibrium outcome.

6 Concluding Remarks

In this paper, we develop a novel framework that addresses how data regulation affects price discrimination. Specifically, we explore the impact of allowing consumers to choose which market to enter before the monopolist sets prices. We fully characterize all possible welfare consequences: The producer surplus remains fixed at the uniform monopoly level, and the consumer surplus can take any value between the uniform monopoly and buyer-optimal levels. More importantly, we find that all possible outcomes are Pareto-improving compared to a scenario where price discrimination is prohibited. Our analysis generates useful theoretical and practical implications, showing that decentralized interactions can achieve the same outcome as centralized manipulation. This suggests that traditional antitrust tools may not be necessary for regulating online monopolies, which starkly contrasts economic theory and conventional wisdom in traditional industries.

We comprehensively investigate the decentralized formation of market segmentation

[35]See https://www.gov.uk/government/collections/algorithmic-transparency-standard for regulations on algorithmic transparency in the United Kingdom.

[36]Tencent, the largest digital enterprise in China and also the largest game company worldwide, derives the majority of its revenue from the games and entertainment sector. Hence, it is reasonable to believe that WeChat, the Tencent instant messenger service, is altruistic to benefit both sellers and buyers.

with up-to-date data protection regulations. However, we formalize the problem in a canonical way with several simplifications. Thus, the current research is merely the first step in studying individual-driven market segmentation. Our approach can also be applied to imperfect circulated markets, where some markets may be unavailable for consumers due to the prohibition on fake information. Consequently, our framework should further capture the topology of different markets, which seems daunting to obtain neat results as in this paper. Another possibility is that only a fraction of consumers can edit their tags, or consumers may be either naive or sophisticated. The relaxation of full information and full rationality may also be considered in future studies. Furthermore, the same question answered in this paper can be duplicated in oligopoly environments. These problems are important for studying price discrimination in E-commerce and can shed light on emerging decentralized autonomous organizations. Therefore, the decentralized market segmentation problem formalized in this paper deserves further exploration.

A Appendix

This Appendix collects the omitted proofs.

Proof of Observation 1. Let $\pi(v|\mathbf{x})$ denote the revenue of pricing v in market $\mathbf{x}$, namely, $\pi(v|\mathbf{x}) \triangleq v\sum_{v_i \geq v} x_i$. Then, $\pi(v|\mathbf{x}+\varepsilon\mathbf{e}_k) = \pi(v|\mathbf{x}) + \mathbb{I}(v \leq v_k)\varepsilon v \leq \pi(v|\mathbf{x}) + \varepsilon v$, where $\mathbb{I}(v \leq v_k)$ equals one if $v \leq v_k$ and equals zero otherwise.

Case I. If $v_k < \phi^{\min}(\mathbf{x})$, then for any $v < \phi^{\min}(\mathbf{x})$,

$$\lim_{\varepsilon\to 0^+} \pi(v|\mathbf{x}+\varepsilon\mathbf{e}_k) \leq \pi(v|\mathbf{x}) + \lim_{\varepsilon\to 0^+} \varepsilon v < \pi(\phi^{\min}(\mathbf{x})|\mathbf{x}) \leq \lim_{\varepsilon\to 0^+} \pi(\phi^{\min}(\mathbf{x})|\mathbf{x}+\varepsilon\mathbf{e}_k).$$

The inequality holds by $\pi(v|\mathbf{x}) < \pi(\phi^{\min}(\mathbf{x})|\mathbf{x})$. Hence, $\lim_{\varepsilon\to 0^+} \phi^{\min}(\mathbf{x}+\varepsilon\mathbf{e}_k) = \phi^{\min}(\mathbf{x})$.

Case II. If $v_k \geq \phi^{\min}(\mathbf{x})$, we assume $\hat{v}$ to be the greatest optimal price in $\mathbf{x}$ that is no larger than v_k. For any price $v \in (\hat{v}, v_k]$,

$$\lim_{\varepsilon\to 0^+} \pi(v|\mathbf{x}+\varepsilon\mathbf{e}_k) = \pi(v|\mathbf{x}) + \lim_{\varepsilon\to 0^+} \varepsilon v < \pi(\hat{v}|\mathbf{x}) = \lim_{\varepsilon\to 0^+} \pi(\hat{v}|\mathbf{x}+\varepsilon\mathbf{e}_k)$$

because $\pi(v|\mathbf{x}) < \pi(\hat{v}|\mathbf{x})$. For any price $v > v_k$, $\pi(v|\mathbf{x}+\varepsilon\mathbf{e}_k) = \pi(v|\mathbf{x}) \leq \pi(\hat{v}|\mathbf{x})$. For any price $v < \hat{v}$, $\pi(v|\mathbf{x}+\varepsilon\mathbf{e}_k) = \pi(v|\mathbf{x}) + \varepsilon v < \pi(\hat{v}|\mathbf{x}) + \varepsilon\hat{v}$. Thus, $\lim_{\varepsilon\to 0^+} \phi^{\min}(\mathbf{x}+\varepsilon\mathbf{e}_k) = \hat{v}$. □

Proof of Lemma 2. Consider two markets $\mathbf{x}_i$ and $\mathbf{x}_j$ in a (weak-)stable segmentation $\sigma(\mathbf{x}^*)$ with $\phi^{\min}(\mathbf{x}_i) < \phi^{\min}(\mathbf{x}_j)$. Apparently, consumers with value $\phi^{\min}(\mathbf{x}_j)$ exist in $\mathbf{x}_j$. Consider a small group of consumers with valuation $\phi^{\min}(\mathbf{x}_j)$ in $\mathbf{x}_j$, if he deviates to market

$\mathbf{x}_i$, $\lim_{\varepsilon\to 0^+} \phi^{\min}(\mathbf{x}_i + \varepsilon \mathbf{e}_k) = \phi^{\min}(\mathbf{x}_j)$ must hold, where $v_k = \phi^{\min}(\mathbf{x}_j)$. Directly from Observation 1, $\phi(\mathbf{x}_j)$ is optimal in market $\mathbf{x}_i$. □

Proof of Lemma 3. Without loss of generality, we assume $\phi^{\min}(\mathbf{x}_t) = \max\left\{\phi^{\min}(\mathbf{x}_i)\right\}_{i=1}^{t}$. Since $\phi^{\min}(\mathbf{x}_t)$ is optimal in all markets within segmentation $\sigma(\mathbf{x}^*)$, the producer surplus is unchanged if charging $\phi^{\min}(\mathbf{x}_t)$ uniformly to all markets. However, charging a uniform price cannot reach a surplus larger than π^*. Therefore, π is fixed at π^* for any stable segmentation by Lemma 1, and $\phi^{\min}(\mathbf{x}_t)$ should be an optimal uniform monopoly price.

Since $\phi^{\min}(\mathbf{x}_t)$ is an optimal uniform monopoly price, $\phi^{\min}(\mathbf{x}_t) \geq \phi^{\min}(\mathbf{x}) = v^*$. Meanwhile, v^* must be optimal in all markets within a stable segmentation. Then, $\phi^{\min}(\mathbf{x}_t) \leq v^*$ because $\phi^{\min}(\mathbf{x}_t)$ is the minimum optimal price in $\mathbf{x}_t$. Therefore, $\phi^{\min}(\mathbf{x}_t) = v^*$. □

Proof of Proposition 1. For necessity, suppose $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$ is a weak-stable segmentation. If $\phi^{\min}(\mathbf{x}_i) < \phi^{\min}(\mathbf{x}_j)$ for markets $\mathbf{x}_i$ and $\mathbf{x}_j$, for a small group of consumers in market $\mathbf{x}_j$ whose value v_k lies in $\left(\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_j)\right]$, his utility is zero. Following the same logic of proving Lemma 2, it should be not profitable for him to deviate to $\mathbf{x}_i$, implying $\lim_{\varepsilon\to 0^+} \phi^{\min}(\mathbf{x}_i + \varepsilon \mathbf{e}_k) = v_k$. By Observation 1, v_k is optimal in $\mathbf{x}_i$.

For sufficiency, suppose segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$ satisfies the condition. For any small group of consumers with value v_k in market $\mathbf{x}_j$, we need to evaluate their deviation to another market $\mathbf{x}_i$. Observation 1 implies that the price cannot decrease after an entry. Hence,

- If $v_k \leq \phi^{\min}(\mathbf{x}_i)$, the price after the entry, $\lim_{\varepsilon\to 0^+} \phi^{\min}(\mathbf{x}_i + \varepsilon \mathbf{e}_k) \geq \phi^{\min}(\mathbf{x}_i) \geq v_k$.
- If $v_k > \phi^{\min}(\mathbf{x}_i)$ but $\phi^{\min}(\mathbf{x}_i) \geq \phi^{\min}(\mathbf{x}_j)$, the price after the entry, $\lim_{\varepsilon\to 0^+} \phi^{\min}(\mathbf{x}_i + \varepsilon \mathbf{e}_k) \geq \phi^{\min}(\mathbf{x}_i) \geq \phi^{\min}(\mathbf{x}_j)$.

Therefore, we suffice to consider the possibilities of deviating to a market $\mathbf{x}_i$ satisfying $\phi^{\min}(\mathbf{x}_i) < \phi^{\min}(\mathbf{x}_j)$ and $v_k > \phi^{\min}(\mathbf{x}_i)$.

When $v_k \geq \phi^{\min}(\mathbf{x}_j)$, since $\phi^{\min}(\mathbf{x}_j) \in \mathbf{supp}\{\mathbf{x}_j\} \cap \left(\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_j)\right]$, $\phi^{\min}(\mathbf{x}_j)$ is optimal in market $\mathbf{x}_i$ by assumption. Then, the price of $\mathbf{x}_i$ after the entry is at least $\phi^{\min}(\mathbf{x}_j)$, indicating no improvement for this consumer. When $v_k < \phi^{\min}(\mathbf{x}_j)$, similarly, the price of $\mathbf{x}_i$ after the entry is at least v_k. Therefore, $\sigma(\mathbf{x}^*)$ is weak-stable. □

Proof of Proposition 2. The necessity is valid by definition.

For sufficiency, we realize that any segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$ satisfy the **no-inflow** condition must be stable. By Lemma 3, the maximum price over markets in $\sigma(\mathbf{x}^*)$ is v^*, and v^* must be optimal in all markets within $\sigma(\mathbf{x}^*)$.

By contradiction, we assume $\sigma(\mathbf{x}^*)$ is not stable and a feasible group deviation $\Big\{\mathbf{x}_i \to (\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')\Big\}_{i=1}^{t}$ exists, where $\mathbf{y}_1 + \cdots + \mathbf{y}_t = \mathbf{y}_1' + \cdots + \mathbf{y}_t' \neq \mathbf{0}$. Let $y_{i,k}$ and $y'_{i,k}$ denote the mass of consumers with v_k in $\mathbf{y}_i$ and $\mathbf{y}_i'$, respectively. Hence, $\Delta y_{i,k} \triangleq y'_{i,k} - y_{i,k}$ denotes the change of consumers with value v_k in market $\mathbf{x}_i$.

Part 1. We first claim that there exists a market $\mathbf{x}_i \in \sigma(\mathbf{x}^*)$ such that

$$\sum_{k:v_k \in \left[\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i'), v^*\right)} \Delta y_{i,k} > 0 \quad \text{and} \quad \sum_{k:v_k \geq v^*} \Delta y_{i,k} \geq 0.$$

The former condition implies that the mass of consumers with value in $[\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i'), v^*)$ strictly increase, where $\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')$ denotes the price in $\mathbf{x}_i$ after the deviation of $\mathbf{y}$; while consumers with value in $[v^*, v_K]$ weakly increase.

We now prove the above statement. Since $\sum_{i=1}^{t}\left(\sum_{k:v_k \geq v^*} \Delta y_{i,k}\right) = 0$, there are two cases: $\sum_{k:v_k \geq v^*} \Delta y_{i,k} = 0$ for all markets; or $\sum_{k:v_k \geq v^*} \Delta y_{i,k} > 0$ for some market $\mathbf{x}_i$.

Case I. Let $\arg\min_{\mathbf{x}_j:\mathbf{y}_j \neq \mathbf{0}} \phi^{\min}(\mathbf{x}_j)$ denote the market with the lowest price among markets with outgoing consumers. Hence, there must exist a market $\mathbf{x}_i$ that has some incoming consumers from the market whose original price is $\arg\min_{\mathbf{x}_j:\mathbf{y}_j \neq \mathbf{0}} \phi^{\min}(\mathbf{x}_j)$, implying $\mathbf{y}_i' \neq \mathbf{0}$. Immediately, we must have $\mathbf{y}_i \neq \mathbf{0}$; otherwise, if we view $\mathbf{y}_i'$ itself as a deviation group, it is a feasible inflow deviation and consumers in $\mathbf{y}_i'$ are strictly better off, which contradicts to the no-inflow condition. Hence, $\mathbf{x}_i$ contains outgoing consumers, and thus $\min_{\mathbf{x}_j:\mathbf{y}_j \neq \mathbf{0}} \phi^{\min}(\mathbf{x}_j) \leq \phi^{\min}(\mathbf{x}_i)$. Since $\mathbf{y}_i'$ contains consumers whose original price is $\min_{\mathbf{x}_j:\mathbf{y}_j \neq \mathbf{0}} \phi^{\min}(\mathbf{x}_j)$, the price faced by $\mathbf{y}_i'$ after the deviation, namely $\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')$, should be strictly lower: $\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i') < \min_{\mathbf{x}_j:\mathbf{y}_j \neq \mathbf{0}} \phi^{\min}(\mathbf{x}_j) \leq \phi^{\min}(\mathbf{x}_i)$.

Since $\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i') < \phi^{\min}(\mathbf{x}_i) \leq v^*$ and $\sum_{v_k \geq v*} \Delta y_{i,k} = 0$, then

$$\begin{aligned}
\sum_{k:v_k \in \left[\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i'), v^*\right)} \Delta y_{i,k} \quad &= \sum_{k:v_k \geq \phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')} \Delta y_{i,k} \\
&= \frac{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i') \sum_{k:v_k \geq \phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')} (x_{i,k} + \Delta y_{i,k})}{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')} - \sum_{k:v_k \geq \phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')} x_{i,k} \\
&\geq \frac{v^* \sum_{k:v_k \geq v^*} (x_{i,k} + \Delta y_{i,k})}{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')} - \sum_{k:v_k \geq \phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')} x_{i,k} \\
&= \frac{v^* \sum_{k:v_k \geq v^*} x_{i,k} - \phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i') \sum_{k:v_k \geq \phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')} x_{i,k}}{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')}
\end{aligned}$$

The inequality holds because pricing $\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i')$ in $\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}_i'$ is no worse than pricing

v^*. Moreover, v^* is optimal in $\mathbf{x}_i$ and hence it is strictly better than $\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)$, $\phi^{\min}(\mathbf{x}_i-\mathbf{y}_i+\mathbf{y}'_i)\sum_{v_k\geq\phi^{\min}(\mathbf{x}_i-\mathbf{y}_i+\mathbf{y}'_i)} x_{i,k} < v^*\sum_{v_k\geq v^*} x_{i,k}$, since $\phi^{\min}(\mathbf{x}_i-\mathbf{y}_i+\mathbf{y}'_i) < \phi^{\min}(\mathbf{x}_i)$. Therefore, we obtain $\sum_{k:v_k\in\left[\phi^{\min}(\mathbf{x}_i-\mathbf{y}_i+\mathbf{y}'_i),v^*\right)} \Delta y_{i,k} > 0$.

Case II. Consider a market $\mathbf{x}_i$ satisfying $\sum_{k:v_k\geq v^*} \Delta y_{i,k} > 0$. Since prices in all markets before the deviation is at most v^*, $\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i) < v^*$ to make those consumers in $\mathbf{y}'_i$ have a strictly higher utility. Since v^* is optimal in $\mathbf{x}_i$ and $\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i) < v^*$ is optimal in $\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i$. Thus, we have

$$
\begin{aligned}
&\sum_{k:v_k\in\left[\phi^{\min}(\mathbf{x}_i-\mathbf{y}_i+\mathbf{y}'_i),v^*\right)} \Delta y_{i,k} \quad = \sum_{k:v_k\geq\phi^{\min}(\mathbf{x}_i-\mathbf{y}_i+\mathbf{y}'_i)} \Delta y_{i,k} - \sum_{k:v_k\geq v^*} \Delta y_{i,k} \\
= \;& \frac{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)\sum_{k:v_k\geq\phi^{\min}(\mathbf{x}_i-\mathbf{y}_i+\mathbf{y}'_i)}(x_{i,k}+\Delta y_{i,k})}{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)} - \sum_{k:v_k\geq v^*} \Delta y_{i,k} - \sum_{k:v_k\geq\phi^{\min}(\mathbf{x}_i-\mathbf{y}_i+\mathbf{y}'_i)} x_{i,k} \\
\geq \;& \frac{v^*\sum_{k:v_k\geq v^*}(x_{i,k}+\Delta y_{i,k})}{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)} - \sum_{k:v_k\geq v^*} \Delta y_{i,k} - \frac{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)\sum_{k:v_k\geq\phi^{\min}(\mathbf{x}_i-\mathbf{y}_i+\mathbf{y}'_i)} x_{i,k}}{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)} \\
\geq \;& \frac{v^*\sum_{k:v_k\geq v^*}(x_{i,k}+\Delta y_{i,k})}{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)} - \sum_{k:v_k\geq v^*} \Delta y_{i,k} - \frac{v^*\sum_{k:v_k\geq v^*} x_{i,k}}{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)} \\
= \;& \left(\frac{v^*}{\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)} - 1\right)\sum_{k:v_k\geq v^*} \Delta y_{i,k} > 0
\end{aligned}
$$

The first inequality holds because pricing $\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)$ in $\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i$ is no worse than pricing v^*. The second inequality holds because pricing v^* in $\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i$ is no worse than pricing $\phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)$.

Part 2. We start with the market $\mathbf{x}_i$ satisfying

$$
\sum_{m:v_m\in\left[\phi^{\min}(\mathbf{x}_i-\mathbf{y}_i+\mathbf{y}'_i),v^*\right)} \Delta y_{i,m} > 0 \quad \text{and} \quad \sum_{m:v_m\geq v^*} \Delta y_{i,m} \geq 0.
$$

We now construct a feasible group deviation that $\mathbf{x}_i$ only has incoming consumers and other markets only have outgoing consumers by modifying $\left\{\mathbf{x}_i \to (\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)\right\}_{i=1}^{t}$. For simplicity, let $v = \phi^{\min}(\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i)$. Our construction has the following three steps and the deviation group shrinks in every single update process with each step.

Step 1. Consider v_k and v_l with $v \leq v_k \leq v_l < v^*$ such that $y_{i,k} > 0, y'_{i,l} > 0$. Let $a = \min\{y_{i,k}, y'_{i,l}\}$. Consider the following update

$$
y_{i,k} \leftarrow y_{i,k} - a, \quad \text{and} \quad y'_{i,l} \leftarrow y'_{i,l} - a, \qquad \text{(Update A)}
$$

which retains a consumers with v_k from $\mathbf{y}_i$ and repatriates a consumers with v_l from $\mathbf{y}'_i$. The revenue of pricing $p \notin (v_k, v_l]$ will remain unchanged and the revenue of pricing $p \in (v_k, v_l]$ will decrease. Then, v is still the minimum optimal price after the update.

Repeat Step 1. Recursively do **Step 1** until there exists no v_k and v_l with $v \leq v_k \leq v_l < v^*$ such that $y_{i,k} > 0$ and $y'_{i,l} > 0$.

Step 2. Consider all v_k and v_l with $v \leq v_k \leq v_l < v^*$ such that $y'_{i,k} > 0, y_{i,l} > 0$. Since the case of $v_k = v_l$ is already resolved by **Repeat Step 1**, we need only to consider the situations where $v \leq v_k < v_l < v^*$ and $y'_{i,k} > 0, y_{i,l} > 0$. Choose a pair of (v_k, v_l) such that $l - k$ is minimized. Then, $l - k > 0$, and $y'_{i,m} = 0$ for $v_m \in (v_k, v^*]$ and $y_{i,m} = 0$ for $v_m \in [v, v_l)$.

- By **Repeat Step 1**, $y'_{i,m} = 0$ for $v_m \in [v_l, v^*]$ and $y_{i,m} = 0$ for $v_m \in [v, v_k]$;
- Since $l - k > 0$ is minimized, $y_{i,m} = 0$ and $y'_{i,m} = 0$ for $v_m \in (v_k, v_l)$.

Let $b = \min\{y'_{i,k}, y_{i,l}\}$. Consider the following update

$$y'_{i,k} \leftarrow y'_{i,k} - b, \quad \text{and} \quad y_{i,l} \leftarrow y_{i,l} - b, \tag{Update B}$$

which repatriates b consumers with v_k from $\mathbf{y}'_i$ and retain b consumers with v_l from $\mathbf{y}_i$. We need to show that the minimum optimal price of the updated market is still v. The revenue of pricing $p \notin (v_k, v_l]$ will remains unchanged. The revenue of pricing $p \in (v_k, v_l]$ after the update is

$$\begin{aligned} p \sum_{m:v_m \geq p} (x_{i,m} + \Delta y_{i,m}) + pb &= p \sum_{m:v_m \geq p} x_{i,m} + p \sum_{m:v_m \geq v^*} \Delta y_{i,m} + p \left(\sum_{m:v_m \in [p,v^*)} \Delta y_{i,m} + b \right) \\ &\leq p \sum_{m:v_m \geq p} x_{i,m} + p \sum_{m:v_m \geq v^*} \Delta y_{i,m} \\ &\leq v^* \sum_{m:v_m \geq v^*} x_{i,m} + v^* \sum_{m:v_m \geq v^*} \Delta y_{i,m} \\ &\leq v \sum_{m:v_m \geq v} (x_{i,m} + \Delta y_{i,m}) \end{aligned}$$

The first inequality holds because

$$\sum_{m:v_m \in [p,v^*)} \Delta y_{i,m} + b = - \sum_{m:v_m \in [p,v^*)} y_{i,m} + b \leq -y_{i,l} + b \leq 0.$$

The second inequality holds because pricing v^* is optimal in $\mathbf{x}_i$ and $p < v^*$. The third inequality holds because pricing v is optimal in $\mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i$.

Repeat Step 2. Recursively do **Step 2** until there exists no v_k and v_l with $v \leq v_k \leq v_l < v^*$ such that $y'_{i,k} > 0$ and $y_{i,l} > 0$.

Step 3. Note that $\sum_{m:v_m \in [v,v^*)} \Delta y_{i,m} > 0$ holds for the original deviation group, and each round of update in **Step 1** or **Step 2** will not change the value of $\sum_{m:v_m \in [v,v^*)} \Delta y_{i,m}$. Hence, the final market after many rounds of updates will have $\sum_{m:v_m \in [v,v^*)} y'_{i,m} > 0$ and $y_{i,m} = 0$ for all $v_m \in [v, v^*)$. Consider another deviation group merely including all consumers associated with $\sum_{m:v_m \in [v,v^*)} y'_{i,m}$. The unique inflow market is $\mathbf{x}_i$ in this deviation group, while all other markets only have outgoing consumers. This deviation group is denoted by $\mathbf{z}$. We claim that all consumers in $\mathbf{z}$ are better off by entering market $\mathbf{x}_i$. Equivalently, since $\mathbf{supp}\{\mathbf{z}\} \in (v, v^*)$, we need to show that $\phi^{\min}(\mathbf{x}_i + \mathbf{z}) \leq v$.[37]

Since v is optimal in the updated market after **Repeat Step 2**, we obtain that pricing v is better than pricing any $v' \in (v, v^*]$ in the ith market after **Step 3** for the following reason. If we compare the updated market after **Repeat Step 2** and the market after **Step 3**, we will find that

- the revenue of pricing some $v' \in (v, v^*]$ is lower by $v' \sum_{k:v_k \geq v^*} \Delta y_{i,k}$;
- the revenue of pricing v is lower by $v \sum_{k:v_k \geq v^*} \Delta y_{i,k}$.

Meanwhile, pricing $v' > v^*$ in $\mathbf{x}_i + \mathbf{z}_i$ is no better than v^* since v^* is optimal in $\mathbf{x}_i$ and no consumer in $\mathbf{z}_i$ has a value in $[v^*, v_K]$. Therefore, $\phi^{\min}(\mathbf{x}_i + \mathbf{z}) \leq v$. $\square$

Proof of Proposition 3. (2) $\Rightarrow$ (1) $\Rightarrow$ (3) holds. To prove (3) $\Rightarrow$ (2), suppose $\phi^{\min}(\mathbf{x}_1) \leq \cdots \leq \phi^{\min}(\mathbf{x}_t)$. By social optimum, $\phi^{\min}(\mathbf{x}_i) = \min \mathbf{supp}\{\mathbf{x}_i\}$. For deviation group $\mathbf{y}$, let i denote the minimum index of those changed markets, and suppose the change is $\mathbf{x}_i \to \mathbf{x}_i - \mathbf{y}_i + \mathbf{y}'_i$. Following the same logic when proving constructed segmentation to be stable, $\mathbf{y}_i = \mathbf{0}$. Then consider consumers with value $\min \mathbf{supp}\{\mathbf{y}'_i\}$ in $\mathbf{y}'_i$. Some of them are originally in market $\mathbf{x}_j$ where $j \geq i$. The price of $\mathbf{x}_i$ after this deviation must be at least $\phi^{\min}(\mathbf{x}_j)$:

- If $\phi^{\min}(\mathbf{x}_j) = \phi^{\min}(\mathbf{x}_i)$, since all consumers in $\mathbf{y}'_i$ have valuations at least $\phi^{\min}(\mathbf{x}_j) = \phi^{\min}(\mathbf{x}_i)$ by social optimum, the price after the entry is at least $\phi^{\min}(\mathbf{x}_j) = \phi^{\min}(\mathbf{x}_i)$.
- If $\phi^{\min}(\mathbf{x}_j) > \phi^{\min}(\mathbf{x}_i)$, then $\phi^{\min}(\mathbf{x}_j)$ should be optimal in $\mathbf{x}_i$ by (3). Pricing $\phi^{\min}(\mathbf{x}_j)$ is strictly better than $\phi^{\min}(\mathbf{x}_i)$ in market $\mathbf{x}_i + \mathbf{y}'_i$.

Therefore, no such group $\mathbf{y}$ exists; $\sigma(\mathbf{x}^*)$ is stable. $\square$

Proof of Proposition 4. We need to verify that $\sigma'(\mathbf{x}^*)$ meets the indifference condition in

[37] Note that $\mathbf{z}_i$ contains part of consumers in $\mathbf{y}'_i$; thus, all consumers in $\mathbf{y}'_i$ has a value larger than v. Hence, $\mathbf{supp}\{\mathbf{z}\} \in [v, v^*)$ by definition but $v \notin \mathbf{supp}\{\mathbf{z}\}$.

Proposition 1. Since $\phi^{\min}(\mathbf{x}_i+\mathbf{x}_j) = \phi^{\min}(\mathbf{x}_i) = \phi^{\min}(\mathbf{x}_j)$, we need to enumerate all markets with a different price. Let $\hat{v} = \phi^{\min}(\mathbf{x}_i+\mathbf{x}_j)$ for brevity. Consider such a market $\mathbf{x}_k \in \sigma(\mathbf{x}^*)$ with a different price, $\phi^{\min}(\mathbf{x}_k) \neq \hat{v}$:

1. $\phi^{\min}(\mathbf{x}_k) > \hat{v}$. Consumers in $\mathbf{x}_k$ have no incentive to enter $\mathbf{x}_i + \mathbf{x}_j$ for the following reason. For all $v \in \mathbf{supp}\{\phi^{\min}(\mathbf{x}_k)\} \cap \left(\hat{v}, \phi^{\min}(\mathbf{x}_k)\right]$, v is also optimal in market $\mathbf{x}_i$ and $\mathbf{x}_j$. Thus, v is also optimal in market $\mathbf{x}_i + \mathbf{x}_j$.
2. $\phi^{\min}(\mathbf{x}_k) < \hat{v}$. Consumers in $\mathbf{x}_i + \mathbf{x}_j$ has no incentive to enter $\mathbf{x}_k$. Since the original segmentation is stable, (i) $\forall v \in \mathbf{supp}\{\mathbf{x}_i\} \cap \left(\phi^{\min}(\mathbf{x}_k), \hat{v}\right]$, v is optimal in market $\mathbf{x}_k$; meanwhile, (ii) $\forall v \in \mathbf{supp}\{\mathbf{x}_j\} \cap \left(\phi^{\min}(\mathbf{x}_k), \hat{v}\right]$, v is optimal in market $\mathbf{x}_k$. Therefore, for all $v \in \mathbf{supp}\{\mathbf{x}_i + \mathbf{x}_j\} \cap \left(\phi^{\min}(\mathbf{x}_k), \hat{v}\right]$, v is also optimal in market $\mathbf{x}_k$.

□

Proof of Proposition 5. Suppose we have two different direct stable segmentations, $\sigma(\mathbf{x}^*)$ and $\sigma'(\mathbf{x}^*)$, with different price profiles. Let $\mathbf{x}_i$ represent a market in σ and $\mathbf{x}'_i$ represent a market in σ'. Without loss of generality, we assume markets are ordered by their prices, $\phi^{\min}(\mathbf{x}_i) < \phi^{\min}(\mathbf{x}_{i+1})$ and $\phi^{\min}(\mathbf{x}'_i) < \phi^{\min}(\mathbf{x}'_{i+1})$. Suppose $\phi^{\min}(\mathbf{x}_{k+1}) > \phi^{\min}(\mathbf{x}'_{k+1})$ and $\phi^{\min}(\mathbf{x}_i) = \phi^{\min}(\mathbf{x}'_i)$ for all $i \leq k$. Evidently, $k \geq 1$ since $\phi^{\min}(\mathbf{x}_1) = \phi^{\min}(\mathbf{x}'_1) = v_1$.

For brevity, we denote $x[a, b) = \sum_{m: v_m \in [a,b)} x_m$ as the mass of consumers in market $\mathbf{x}$ with value in the interval $[a, b)$. Let π_i and π'_i denote the revenue of the producer in market $\mathbf{x}_i$ and $\mathbf{x}'_i$, respectively. For any market $\mathbf{x}_i$, by Proposition 3, we have an indifference relation because $\phi^{\min}(\mathbf{x}_{i+1})$ should be optimal in $\mathbf{x}_i$: $\phi^{\min}(\mathbf{x}_i)x_i[\phi^{\min}(\mathbf{x}_i), \infty) = \phi^{\min}(\mathbf{x}_{i+1})x_i[\phi^{\min}(\mathbf{x}_{i+1}), \infty)$. Rearranging as $x_i[\phi^{\min}(\mathbf{x}_{i+1}), \infty) = \frac{\phi^{\min}(\mathbf{x}_i)x_i[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))}{\phi^{\min}(\mathbf{x}_{i+1}) - \phi^{\min}(\mathbf{x}_i)}$, we have

$$\pi_i = \phi^{\min}(\mathbf{x}_{i+1})x_i[\phi^{\min}(\mathbf{x}_{i+1}), \infty) = \frac{\phi^{\min}(\mathbf{x}_i)\phi^{\min}(\mathbf{x}_{i+1})x_i[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))}{\phi^{\min}(\mathbf{x}_{i+1}) - \phi^{\min}(\mathbf{x}_i)} \tag{1}$$

Step 1. Consider market $\mathbf{x}_k$. Notice that $x_k[\phi^{\min}(\mathbf{x}_{k+1}), \infty) = \frac{\phi^{\min}(\mathbf{x}_k)x_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}))}{\phi^{\min}(\mathbf{x}_{k+1}) - \phi^{\min}(\mathbf{x}_k)}$. Since we assume $\phi^{\min}(\mathbf{x}_k) < \phi^{\min}(\mathbf{x}'_{k+1}) < \phi^{\min}(\mathbf{x}_{k+1})$, we obtain

$$x_k[\phi^{\min}(\mathbf{x}_{k+1}), \infty) = \frac{\phi^{\min}(\mathbf{x}_k)\left(x_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}'_{k+1})) + x_k[\phi^{\min}(\mathbf{x}'_{k+1}), \phi^{\min}(\mathbf{x}_{k+1}))\right)}{\phi^{\min}(\mathbf{x}_{k+1}) - \phi^{\min}(\mathbf{x}_k)} \tag{2}$$

Meanwhile $\phi^{\min}(\mathbf{x}'_{k+1})$ cannot realize a higher profit than $\phi^{\min}(\mathbf{x}_{k+1})$ in market $\mathbf{x}_k$: $\phi^{\min}(\mathbf{x}'_{k+1})(x_k[\phi^{\min}(\mathbf{x}'_{k+1}), \phi^{\min}(\mathbf{x}_{k+1})) + x_k[\phi^{\min}(\mathbf{x}_{k+1}), \infty)) \leq \phi^{\min}(\mathbf{x}_{k+1})x_k[\phi^{\min}(\mathbf{x}_{k+1}), \infty)$, which can be rearranged as $x_k[\phi^{\min}(\mathbf{x}'_{k+1}), \phi^{\min}(\mathbf{x}_{k+1})) \leq \frac{(\phi^{\min}(\mathbf{x}_{k+1}) - \phi^{\min}(\mathbf{x}'_{k+1}))x_k[\phi^{\min}(\mathbf{x}_{k+1}), \infty)}{\phi^{\min}(\mathbf{x}'_{k+1})}$.

Replacing $x_k[\phi^{\min}(\mathbf{x}_{k+1}), \infty)$ by Equation 2, we have

$$\frac{x_k[\phi^{\min}(\mathbf{x}'_{k+1}), \phi^{\min}(\mathbf{x}_{k+1}))}{x_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}'_{k+1}))} \leq \frac{\phi^{\min}(\mathbf{x}_k)(\phi^{\min}(\mathbf{x}_{k+1}) - \phi^{\min}(\mathbf{x}'_{k+1}))}{\phi^{\min}(\mathbf{x}_{k+1})(\phi^{\min}(\mathbf{x}'_{k+1}) - \phi^{\min}(\mathbf{x}_k))} \tag{3}$$

Consider market $\mathbf{x}'_k$. Similarly, pricing $\mathbf{x}'_{k+1}$ is optimal in $\mathbf{x}'_k$ by Proposition 3, we have the indifference relation: $\phi^{\min}(\mathbf{x}'_k)x'_k[\phi^{\min}(\mathbf{x}'_k), \infty) = \phi^{\min}(\mathbf{x}'_{k+1})x'_k[\phi^{\min}(\mathbf{x}'_{k+1}), \infty)$, which can be rearranged as

$$x'_k[\phi^{\min}(\mathbf{x}'_{k+1}), \infty) = \frac{\phi^{\min}(\mathbf{x}_k)x'_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}'_{k+1}))}{\phi^{\min}(\mathbf{x}'_{k+1}) - \phi^{\min}(\mathbf{x}_k)} \tag{4}$$

since $\phi^{\min}(\mathbf{x}'_k) = \phi^{\min}(\mathbf{x}_k)$. Meanwhile, $\phi^{\min}(\mathbf{x}_{k+1})$ cannot realize a higher profit than $\phi^{\min}(\mathbf{x}'_{k+1})$ in market $\mathbf{x}'_k$: $\phi^{\min}(\mathbf{x}'_{k+1})x'_k[\phi^{\min}(\mathbf{x}'_{k+1}), \infty) \geq \phi^{\min}(\mathbf{x}_{k+1})(x'_k[\phi^{\min}(\mathbf{x}'_{k+1}), \infty) - x'_k[\phi^{\min}(\mathbf{x}'_{k+1}), \phi^{\min}(\mathbf{x}_{k+1})))$. Replacing $x'_k[\phi^{\min}(\mathbf{x}'_{k+1}), \infty)$ by Equation 4, we obtain

$$\frac{x'_k[\phi^{\min}(\mathbf{x}'_{k+1}), \phi^{\min}(\mathbf{x}_{k+1}))}{x'_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}'_{k+1}))} \geq \frac{\phi^{\min}(\mathbf{x}_k)(\phi^{\min}(\mathbf{x}_{k+1}) - \phi^{\min}(\mathbf{x}'_{k+1}))}{\phi^{\min}(\mathbf{x}_{k+1})(\phi^{\min}(\mathbf{x}'_{k+1}) - \phi^{\min}(\mathbf{x}_k))} \tag{5}$$

Equation 3 and Equation 5 are suffice to derive $\frac{x_k[\phi^{\min}(\mathbf{x}'_{k+1}), \phi^{\min}(\mathbf{x}_{k+1}))}{x_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}'_{k+1}))} \leq \frac{x'_k[\phi^{\min}(\mathbf{x}'_{k+1}), \phi^{\min}(\mathbf{x}_{k+1}))}{x'_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}'_{k+1}))}$. Equivalently,

$$\frac{x_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}))}{x_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}'_{k+1}))} \leq \frac{x'_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}))}{x'_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}'_{k+1}))}. \tag{6}$$

Step 2. Let π_i and π'_i denote the revenue of markets $\mathbf{x}_i$ and $\mathbf{x}'_i$, respectively.

The goal here is to show $\pi_i = \pi'_i$ for $i \leq k-1$. If $k = 1$, the statement holds trivially. Hence, we assume $k \geq 2$.

Base Case. By $\phi^{\min}(\mathbf{x}_1) = \phi^{\min}(\mathbf{x}'_1), \phi^{\min}(\mathbf{x}_2) = \phi^{\min}(\mathbf{x}'_2)$ and Equation 1 we have $\pi_1 = \frac{\phi^{\min}(\mathbf{x}_1)\phi^{\min}(\mathbf{x}_2)x_1[\phi^{\min}(\mathbf{x}_1), \phi^{\min}(\mathbf{x}_2))}{\phi^{\min}(\mathbf{x}_2) - \phi^{\min}(\mathbf{x}_1)}$ and $\pi'_1 = \frac{\phi^{\min}(\mathbf{x}_1)\phi^{\min}(\mathbf{x}_2)x'_1[\phi^{\min}(\mathbf{x}_1), \phi^{\min}(\mathbf{x}_2))}{\phi^{\min}(\mathbf{x}_2) - \phi^{\min}(\mathbf{x}_1)}$. Furthermore, $x_1[\phi^{\min}(\mathbf{x}_1), \phi^{\min}(\mathbf{x}_2)) = x^*[\phi^{\min}(\mathbf{x}_1), \phi^{\min}(\mathbf{x}_2)) = x'_1[\phi^{\min}(\mathbf{x}_1), \phi^{\min}(\mathbf{x}_2))$ holds because all consumers whose value in the interval $[\phi^{\min}(\mathbf{x}_1), \phi^{\min}(\mathbf{x}_2))$ must go to $\mathbf{x}_1 \in \sigma(\mathbf{x}^*)$ or $\mathbf{x}'_1 \in \sigma'(\mathbf{x}^*)$ to guarantee social optimum. Therefore, $\pi_1 = \pi'_1$.

Induction. Suppose $\pi_1 = \pi'_1, \cdots, \pi_{i-1} = \pi'_{i-1}$ and consider π_i, π'_i where $i \leq k-1$. We only need to show that $x_i[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1})) = x'_i[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))$. If so, $\pi_i = \pi'_i$ holds because $\phi^{\min}(\mathbf{x}_i) = \phi^{\min}(\mathbf{x}'_i), \phi^{\min}(\mathbf{x}_{i+1}) = \phi^{\min}(\mathbf{x}'_{i+1})$, Equation 1, and

$$\pi_i = \frac{\phi^{\min}(\mathbf{x}_i)\phi^{\min}(\mathbf{x}_{i+1})x_i[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))}{\phi^{\min}(\mathbf{x}_{i+1}) - \phi^{\min}(\mathbf{x}_i)}, \pi'_i = \frac{\phi^{\min}(\mathbf{x}_i)\phi^{\min}(\mathbf{x}_{i+1})x'_i[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))}{\phi^{\min}(\mathbf{x}_{i+1}) - \phi^{\min}(\mathbf{x}_i)}.$$

To show $x_i[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1})) = x_i'[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))$, we realize that the following identity must be valid because every consumer whose value in the interval $[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))$ must go to one market in each segmentation:

$$\underbrace{\sum_{j=1}^{i} x_j[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))}_{\text{All consumers with value } [\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1})) \text{ in } \sigma} = \underbrace{\sum_{j=1}^{i} x_j'[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))}_{\text{All consumers with value } [\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1})) \text{ in } \sigma'} .$$

Since $x_j[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1})) = \frac{\pi_j}{\phi^{\min}(\mathbf{x}_i)} - \frac{\pi_j}{\phi^{\min}(\mathbf{x}_{i+1})} = x_j'[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))$ for $j \le i-1$, $x_i[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1})) = x_i'[\phi^{\min}(\mathbf{x}_i), \phi^{\min}(\mathbf{x}_{i+1}))$ holds immediately.

Step 3. The objective here is to show that $x_k'[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}')) \ge x_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}'))$ and $x_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1})) \ge x_k'[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}))$. We start with an identity,

$$\underbrace{\sum_{j=1}^{k} x_j[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}'))}_{\text{All consumers with value } [\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}')) \text{ in } \sigma} = \underbrace{\sum_{j=1}^{k} x_j'[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}'))}_{\text{All consumers with value } [\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}')) \text{ in } \sigma'} .$$

Meanwhile, $x_j'[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}')) = \frac{\pi_j}{\phi^{\min}(\mathbf{x}_k)} - \frac{\pi_j}{\phi^{\min}(\mathbf{x}_{k+1}')}$ for $j \le k-1$, since both $\phi^{\min}(\mathbf{x}_k)$ and $\phi^{\min}(\mathbf{x}_{k+1}')$ are optimal prices for markets $\{\mathbf{x}_1', \cdots, \mathbf{x}_{k-1}'\}$. Although $\phi^{\min}(\mathbf{x}_k)$ is also optimal price for markets $\{\mathbf{x}_1, \cdots, \mathbf{x}_{k-1}\}$, $\phi^{\min}(\mathbf{x}_{k+1}')$ may not be optimal any more. Hence, $x_j[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}')) \ge \frac{\pi_j}{\phi^{\min}(\mathbf{x}_k)} - \frac{\pi_j}{\phi^{\min}(\mathbf{x}_{k+1}')} = x_j'[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}'))$ for $j \le k-1$, because $\frac{\pi_j}{\phi^{\min}(\mathbf{x}_k)} = x_j[\phi^{\min}(\mathbf{x}_k), \infty)$ and $\frac{\pi_j}{\phi^{\min}(\mathbf{x}_{k+1}')} \ge x_j[\phi^{\min}(\mathbf{x}_{k+1}'), \infty)$. Therefore,

$$x_k'[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}')) \ge x_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}')). \tag{7}$$

Similarly,

$$\underbrace{\sum_{j=1}^{k} x_j[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}))}_{\text{All consumers with value } [\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1})) \text{ in } \sigma} \ge \underbrace{\sum_{j=1}^{k} x_j'[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}))}_{\text{Some consumers with value } [\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1})) \text{ in } \sigma'} .$$

while $x_j'[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1})) \ge \frac{\pi_j}{\phi^{\min}(\mathbf{x}_k)} - \frac{\pi_j}{\phi^{\min}(\mathbf{x}_{k+1})} = x_j[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1}))$ for $j \le k-1$. Therefore,

$$x_k[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1})) \ge x_k'(\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1})). \tag{8}$$

Step 4. By Equation 6, Equation 7, and Equation 8, we realize that all inequalities mentioned above must be equations. In particular,

$$\sum_{j=1}^{k} x_j[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1})) = x^*[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1})) = \sum_{j=1}^{k} x'_j[\phi^{\min}(\mathbf{x}_k), \phi^{\min}(\mathbf{x}_{k+1})),$$

implying that no consumer with value $\phi^{\min}(\mathbf{x}'_{k+1})$ exists in market $\mathbf{x}'_{k+1}$. This fact contradicts the optimum of $\phi^{\min}(\mathbf{x}'_{k+1})$ in $\mathbf{x}'_{k+1}$. Therefore, the price profile of any stable, social-optimal, direct segmentation is the same. The equivalence of revenue profile can be proven by a similar induction method as **Step 2**. □

Step-by-Step Interpretation of Figure 5 and Figure 6

We will briefly explain why Figure 5 and Figure 6 provide the price profile and revenue profile for the greedy segmentation introduced in Section 3.1.

Recall that $\mathbf{x}_1^{\text{Greedy}}$ is an extremal market with $\mathbf{supp}\{\mathbf{x}_1^{\text{Greedy}}\} = V$. Hence, we need to explain geometrically which value is run out of at the first iteration. Recall that $\mathbf{x}^* = \mathbf{x}_1^{\text{Greedy}} + \mathbf{x}^{(1)}$, where $\mathbf{x}^{(1)}$ denotes residual market after generating $\mathbf{x}_1^{\text{Greedy}}$. Some value v_k is exhausted, namely $x_k^{(1)} = 0$, if and only if

$$\frac{\hat{\pi}_{\mathbf{x}^*}(v_{k+1}) - \hat{\pi}_{\mathbf{x}_1^{\text{Greedy}}}(v_{k+1})}{v_{k+1}} = \frac{\hat{\pi}_{\mathbf{x}^*}(v_k) - \hat{\pi}_{\mathbf{x}_1^{\text{Greedy}}}(v_k)}{v_k},$$

because $\frac{\hat{\pi}_{\mathbf{x}^{(1)}}(v_{k+1})}{v_{k+1}} = \sum_{i=k+1}^{K} x_i^{(1)} = \sum_{i=k}^{K} x_i^{(1)} = \frac{\hat{\pi}_{\mathbf{x}^{(1)}}(v_k)}{v_k}$, where $v_{K+1} = \infty$. Hence, in the first round of iteration, the producer's profit in $\mathbf{x}_1^{\text{Greedy}}$ is determined by

$$\max\left\{\pi : \frac{\hat{\pi}_{\mathbf{x}^*}(v_{k+1}) - \pi}{v_{k+1}} \leq \frac{\hat{\pi}_{\mathbf{x}^*}(v_k) - \pi}{v_k}\right\}.$$

Since the extremal market $\mathbf{x}_1$ packs as many consumers as possible with all valuations on V, none of possible valuations are exhausted if $\frac{\hat{\pi}_{\mathbf{x}^*}(v_{k+1})-\pi}{v_{k+1}} < \frac{\hat{\pi}_{\mathbf{x}^*}(v_k)-\pi}{v_k}$ for all k. Let $\pi_k^{(1)}$ solves the equation $\frac{\hat{\pi}_{\mathbf{x}^*}(v_{k+1})-\pi}{v_{k+1}} = \frac{\hat{\pi}_{\mathbf{x}^*}(v_k)-\pi}{v_k}$, where $v_k \in \mathbf{supp}\{\mathbf{x}^*\}$. Hence, the exhausted value is equivalent to finding the minimum $\pi_k^{(1)}$ among $k = 1, \cdots, K$. Since $(0, \pi_k^{(1)})$, $(v_k, \hat{\pi}_{\mathbf{x}^*}(v_k))$ and $(v_{k+1}, \hat{\pi}_{\mathbf{x}^*}(v_{k+1}))$ are collinear, $\pi_k^{(1)}$ is the vertical intercept of the line connecting $(v_k, \hat{\pi}_{\mathbf{x}^*}(v_k))$ and $(v_{k+1}, \hat{\pi}_{\mathbf{x}^*}(v_{k+1}))$, where $(v_{K+1}, \hat{\pi}_{\mathbf{x}^*}(v_{K+1})) \equiv (+\infty, 0)$.

Let us see Figure 7 for the first iteration in the greedy procedures on Example 3. We enumerate all five vertical intercepts: four of them are pinned down by extending four line segments, where line segments are plotted by thick lines and extended lines are plotted by dotted lines; the fifth is determined by drawing horizontal line crossing $(v_5, \hat{\pi}_{\mathbf{x}^*}(v_5))$, which can be regarded as extending line segmentation from $(v_5, \hat{\pi}_{\mathbf{x}^*}(v_5))$ to $(+\infty, 0)$. Apparently, the minimum vertical intercept is determined by extending the line segment between $(v_1, \hat{\pi}_{\mathbf{x}^*}(v_1))$ and $(v_2, \hat{\pi}_{\mathbf{x}^*}(v_2))$. As a result and marked by blue, v_1 is exhausted in the first iteration, and the producer's profit in $\mathbf{x}_1^{\text{Greedy}}$ equals this intercept, denoted as π_1^{G}. Using the setup in Example 3, $\pi_1^{\text{G}} = 0.2$. Hence,

$$\mathbf{x}_1^{\text{Greedy}} = \pi_1^{\text{G}} \left(\frac{1}{v_1} - \frac{1}{v_2}, \frac{1}{v_2} - \frac{1}{v_3}, \frac{1}{v_3} - \frac{1}{v_4}, \frac{1}{v_4} - \frac{1}{v_5}, \frac{1}{v_5} \right) = \left(\frac{1}{10}, \frac{1}{30}, \frac{1}{60}, \frac{1}{100}, \frac{1}{25} \right).$$

We can verify that pricing v_1, v_2, v_3, v_4, v_5 is equivalent in $\mathbf{x}_1^{\text{Greedy}}$ and v_1 is exhausted.

Until now, we have established the correctness of generating an extremal market based on the revenue function $\hat{\pi}_{\mathbf{x}^*}$. Therefore, we can generate the extremal market $\mathbf{x}^{(k)}$ based on the corresponding revenue function $\hat{\pi}_{\mathbf{x}^{(k-1)}}$ in each iteration, where $\mathbf{x}^{(0)} = \mathbf{x}^*$. For example, the second extremal market $\mathbf{x}_2^{\text{Greedy}}$ are output based on the revenue function $\hat{\pi}_{\mathbf{x}^{(1)}}$ for the second iteration.

In particular, the revenue function in the kth round is defined as $\hat{\pi}_{\mathbf{x}^{(k-1)}}(v_i) = \hat{\pi}_{\mathbf{x}^*}(v_i) - \sum_{i=1}^{k-1} \pi_i^{\text{G}}$ for $v_i \in \mathbf{supp}\{\mathbf{x}^{(k-1)}\}$, where π_i^{G} denotes the producer's profit on $\mathbf{x}_i^{\text{Greedy}}$. Recall that $\mathbf{x}^* = \sum_{i=1}^{k} \mathbf{x}_i^{\text{Greedy}} + \mathbf{x}^{(k)}$, where $\mathbf{x}^{(k)}$ denotes residual market after generating $\mathbf{x}_k^{\text{Greedy}}$. Some value v_m is exhausted, namely $x_m^{(k)} = 0, v_m \in \mathbf{supp}\{\mathbf{x}^{(k-1)}\}$, if and only if

$$\frac{\hat{\pi}_{\mathbf{x}^*}(v_{m+}) - \sum_{i=1}^{k-1} \pi_i^{\text{G}} - \hat{\pi}_{\mathbf{x}_k^{\text{Greedy}}}(v_{m+})}{v_{m+}} = \frac{\hat{\pi}_{\mathbf{x}^*}(v_m) - \sum_{i=1}^{k-1} \pi_i^{\text{G}} - \hat{\pi}_{\mathbf{x}_m^{\text{Greedy}}}(v_m)}{v_m},$$

where $v_{m+} \equiv \mu(v_m, \mathbf{supp}\{\mathbf{x}^{(k-1)}\})$ denotes the smallest element in $\mathbf{supp}\{\mathbf{x}^{(k-1)}\}$ that higher than v_m. Hence, in the kth round, producer's profit in $\mathbf{x}_k^{\text{Greedy}}$ is determined by

$$\max \left\{ \pi : \frac{\hat{\pi}_{\mathbf{x}^*}(v_{m+}) - \sum_{i=1}^{k-1} \pi_i^{\text{G}} - \pi}{v_{m+}} \leq \frac{\hat{\pi}_{\mathbf{x}^*}(v_m) - \sum_{i=1}^{k-1} \pi_i^{\text{G}} - \pi}{v_m} \right\}.$$

Let $\pi_m^{(k)}$ solves the equation $\frac{\hat{\pi}_{\mathbf{x}^*}(v_{m+}) - \pi_1^{\text{G}} - \pi}{v_{m+}} = \frac{\hat{\pi}_{\mathbf{x}^*}(v_m) - \pi_1^{\text{G}} - \pi}{v_m}$, where $v_m \in \mathbf{supp}\{\mathbf{x}^{(k-1)}\}$. Hence, the exhausted value is equivalent to finding the minimum $\pi_m^{(k)}$. Since $(0, \sum_{i=1}^{k-1} \pi_i^{\text{G}} + \pi_m^{(k)})$, $(v_m, \hat{\pi}_{\mathbf{x}^*}(v_m))$ and $(v_{m+}, \hat{\pi}_{\mathbf{x}^*}(v_{m+}))$ are collinear, $\sum_{i=1}^{k-1} \pi_i^{\text{G}} + \pi_m^{(k)}$ is the vertical intercept of the line connecting $(v_m, \hat{\pi}_{\mathbf{x}^*}(v_m))$ and $(v_{m+}, \hat{\pi}_{\mathbf{x}^*}(v_{m+}))$.

Let us move to the second iteration of Example 3 summarized in Figure 8. Apparently, $\mathbf{supp}\{\mathbf{x}^{(1)}\} = \{v_2, v_3, v_4, v_5\}$. We enumerate four vertical intercepts: three of them are pinned down by extending three line segments, where line segments are plotted by thick lines and extended lines are plotted by dotted lines; the fourth is determined by drawing horizontal line crossing $(v_5, \hat{\pi}_{\mathbf{x}^*}(v_5))$. Then, we will find that the consumers with v_5 are fully spent in the second iteration since the vertical intercept determined associated with v_5 is minimum. The revenue by pricing any valuation in $\mathbf{x}_2^{\text{Greedy}}$ is the same and denoted as π_2^{G}. Using the setup in Example 3, $\pi_1^{\text{G}} + \pi_2^{\text{G}} = 1$ and thus $\pi_2^{\text{G}} = 0.8$. Hence, the second extremal market is

$$\mathbf{x}_2^{\text{Greedy}} = \pi_2^{\text{G}} \left(0, \frac{1}{v_2} - \frac{1}{v_3}, \frac{1}{v_3} - \frac{1}{v_4}, \frac{1}{v_4} - \frac{1}{v_5}, \frac{1}{v_5}\right) = \left(0, \frac{2}{15}, \frac{1}{15}, \frac{1}{25}, \frac{4}{25}\right).$$

Pricing v_2, v_3, v_4, v_5 is equivalent in $\mathbf{x}_2^{\text{Greedy}}$ and v_5 is exhausted. Since consumers with value v_2 remain at the residual market, price in $\mathbf{x}_3^{\text{Greedy}}$ is also expected to be v_2.

In the third iteration, there are only v_2, v_3, v_4 consumers remaining. Continue to find the minimum among three vertical intercepts, we can see that v_3 is exhausted because the minimum is attained by extending the line segment between $(v_3, \hat{\pi}_{\mathbf{x}^*}(v_3))$ and $(v_4, \hat{\pi}_{\mathbf{x}^*}(v_4))$ (see Figure 9). As a result and marked by blue, v_3 is exhausted in the third iteration, and the producer's profit in $\mathbf{x}_3^{\text{Greedy}}$ is denoted as π_3^{G}. Using the setup in Example 3, $\pi_3^{\text{G}} = 0.2$. Hence, the third extremal market is

$$\mathbf{x}_3^{\text{Greedy}} = \pi_3^{\text{G}} \left(0, \frac{1}{v_2} - \frac{1}{v_3}, \frac{1}{v_3} - \frac{1}{v_4}, \frac{1}{v_4}, 0\right) = \left(0, \frac{1}{30}, \frac{1}{60}, \frac{1}{20}, 0\right).$$

We can verify that pricing v_2, v_3, v_4 is equivalent in $\mathbf{x}_3^{\text{Greedy}}$ and v_3 is exhausted. The fourth and fifth rounds are operated similarly, and the graphical illustrations are shown in Figure 10 and Figure 11, respectively. Merging markets with the same price to obtain the price profile and revenue profile shown in Figure 12. Removing all dotted lines to obtain Figure 6.

Proof of Lemma 4. The prices of all markets must be the same. If not, suppose there exist two different markets $\mathbf{x}_i, \mathbf{x}_j \in \sigma(\mathbf{x}^*)$ with different prices, $\phi^{\max}(\mathbf{x}_i) < \phi^{\max}(\mathbf{x}_j)$. Consider the consumers whose valuation is $\phi^{\max}(\mathbf{x}_j)$ in market $\mathbf{x}_j$, which is guaranteed to exist. Since $\mathbf{x}_i$ using the maximum optimal pricing, we argue that $\lim_{\varepsilon \to 0^+} \phi^{\max}(\mathbf{x} + \varepsilon \mathbf{e}_k) = \phi^{\max}(\mathbf{x})$, which is parallel to Observation 1. Thus, the proof is similar. Then, after the deviation, the price will not change. Hence, it is not stable.

All markets must have the same price to be stable. It remains to show that this price

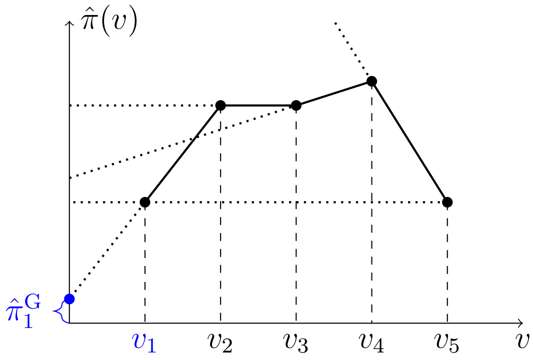

Figure 7: First Iteration: v_1 is exhausted

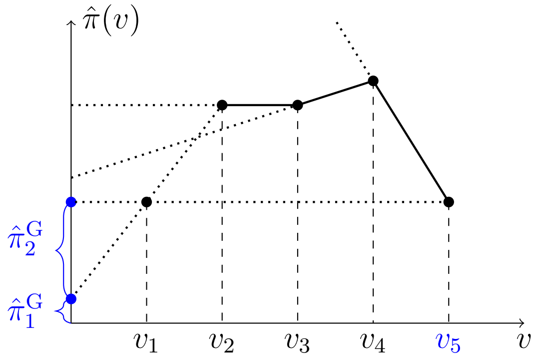

Figure 8: Second Iteration: v_5 is exhausted

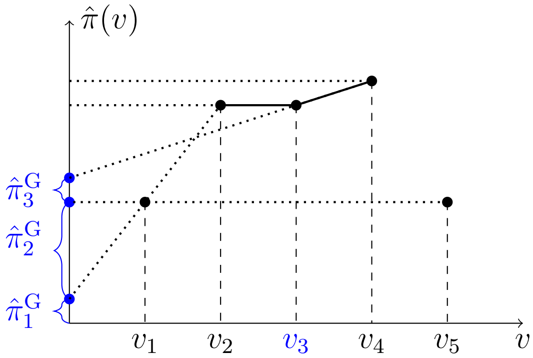

Figure 9: Third Iteration: v_3 is exhausted

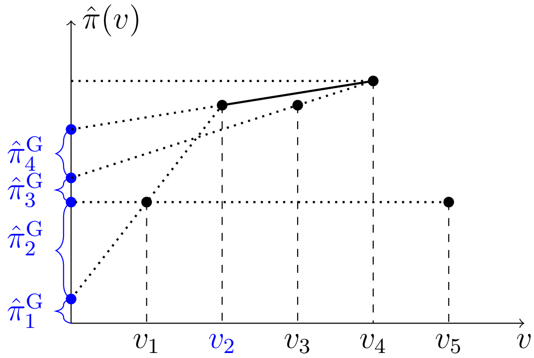

Figure 10: Fourth Iteration: v_2 is exhausted

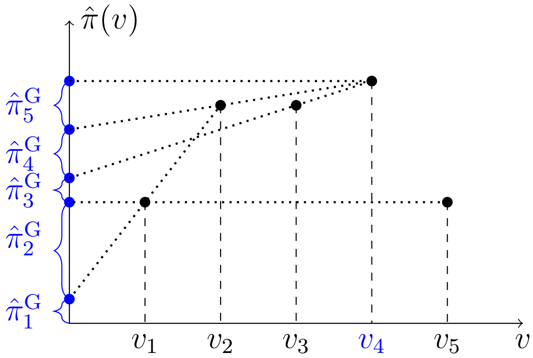

Figure 11: Fifth Iteration: v_4 is exhausted

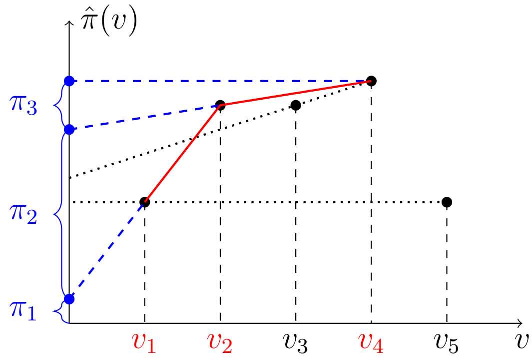

Figure 12: Direct Segmentation

must be the maximal optimal uniform price. Since $\phi^{\max}(\mathbf{x})$ is the same for every $\mathbf{x} \in \sigma(\mathbf{x}^*)$, we conclude that $\phi^{\max}(\mathbf{x}^*) = \phi^{\max}(\sum_{\mathbf{x}\in\sigma(\mathbf{x}^*)} \mathbf{x}) = \phi^{\max}(\mathbf{x})$. □

Proof of Proposition 6. Consider any stable market segmentation $\sigma(\mathbf{x}^*) = \{\mathbf{x}_1, \cdots, \mathbf{x}_t\}$ under the pricing rule ϕ^O. Without loss of generality, we assume $\phi^O(\mathbf{x}_1) \leq \cdots \leq \phi^O(\mathbf{x}_t)$. The merit of this proof is similar to Lemma 3. First, we show that $\phi^O(\mathbf{x}_t)$ is optimal in all markets $\mathbf{x}_i \in \sigma(\mathbf{x}^*)$. Second, we show that $\phi^O(\mathbf{x}_t)$ is optimal uniform price.

Step 1. Consider a group of consumers in market $\mathbf{x}_t$ with valuation $\phi^O(\mathbf{x}_t) = v_t$. This group must exist since any optimal price is supported. If this group of consumers enters

$\mathbf{x}_i$, the revenue in $\mathbf{x}_i + \varepsilon \mathbf{e}_t$ by pricing $v > v_t$ will remain the same compared with market $\mathbf{x}_i$, while the revenue by pricing $v \leq v_t$ strictly increases. This indicates that any $v > v_t$ cannot be optimal in $\mathbf{x}_i + \varepsilon \mathbf{e}_t$. Therefore, the price after the entry, $\phi^O(\mathbf{x}_i + \varepsilon \mathbf{e}_t)$, must be v_t to make the deviation unprofitable. This implies that $\phi^O(\mathbf{x}_t)$ is optimal in $\mathbf{x}_i$.

Step 2. Since $\phi^O(\mathbf{x}_t)$ is optimal in each market, the producer surplus must be π^* by pricing $\phi^O(\mathbf{x}_t)$. This fact holds since π^* is the maximum possible surplus by a uniform price and the minimum possible surplus by a rational pricing rule.

Since $\phi^O(\mathbf{x}_t)$ is at most the maximum optimal uniform monopoly price, the consumer surplus is at least the surplus under the maximum optimal uniform price. □

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