Implementing Randomized Allocation Rules with Outcome-Contingent Transfers*

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Abstract

We study a mechanism design problem where the allocation rule is randomized and transfers are contingent on outcomes. In this problem, an agent reports his private information, and an exogenous randomized allocation rule assigns an outcome based on the report. A planner designs an outcome-contingent transfer to incentivize the agent to report truthfully. We say that the allocation rule is implementable if such transfers exist. For this implementation problem, we derive two sufficient and necessary conditions. Each has a geometric interpretation. Moreover, when the allocation rule is implementable, we construct transfers that implement the allocation rule.

Keywords. Mechanism Design; Implementation; Transfer; Dual Cone; Convex Envelope
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1 Introduction

A central topic in accounting is earnings management. Each year, firms are required to report their annual economic activities to an auditing company in order to compile financial statements. Auditors exercise a certain level of discretion in this task due to reputation and potential legal implications. These financial statements hold significant importance for the firm, as they directly impact future financing costs. Furthermore, taxes are calculated based on these statements. In practice, some firms resort to tactics such as window dressing, which involves inflating their reports to present a more favorable financial picture. Firms may also engage in earning manipulation, deliberately under reporting their performance to minimize tax liabilities. This prompts a natural question: is there a tax regime that could encourage firms to report truthfully? If such a regime exists, what does it look like?

Similar challenges arise in the field of political science. For instance, on an annual basis, each province in China reports its economic growth, fiscal surplus, expected annual budget, and other relevant information to the Bureau of Statistics. This report comprises high-dimensional data, encompassing all aspects of economic activity. The Bureau of Statistics evaluates the overall economic condition of each province. The evaluation is of particular interest to the provinces as it may influence their economic policies in the future.[1] Based on the evaluation, the central government determines the fiscal transfers between provinces.[2] Is it possible for the central government to devise a transfer scheme that incentivizes every province to truthfully report its economic condition?

We analyze these problems using a mechanism design approach. We study a model where an agent knows the underlying state of the world, θ, which belongs to a set Θ. The agent reports a state to an exogenous *allocation rule*, a function $\pi\colon \Theta \to \Delta(X)$ mapping each state to a distribution over a finite outcome space X. The agent's valuation of the outcome x in state θ is given by $v(\theta, x)$. A planner designs an outcome-contingent transfer $t\colon X \to \mathbb{R}$ describing the agent's monetary

[1]For example, many of China's economic special zones and new areas are selected due to fast economic growth, including Shenzhen Special Economic Zone, Shanghai Pudong New Area, and Zhuhai Hengqin New Area in Guangdong. Once established as special economic zones or new areas, a district enjoys special policy treatment, including tax reduction, relaxation of market access, simplification of administrative approval, etc.

[2]The magnitude of transfer for each province is roughly ten billion dollars. The aggregate transfer is roughly a trillion dollars.

payoff as a function of the outcome. Our research question is whether there exists an outcome-contingent transfer to induce the agent to report the state truthfully.

In the earnings management example, the agent is a firm that reports its financial situation θ to an auditing company. The company's auditing produces a financial statement x. The allocation rule summarizes the auditor's practices and protocol. Then the government collects tax $t(x)$ based on the financial statement. In the fiscal transfer example, the agent is a province that reports its economic activities θ to the Bureau of Statistics. The Bureau of Statistics assigns an economic evaluation x to the province. The allocation rule summarizes the evaluation procedure. (The randomness in the evaluation rule is to reduce a province's incentive to manipulate its report.) Then the central government assigns a transfer $t(x)$ based on the evaluation.[3]

In these two examples, the transfer's contingency on the evaluation (financial statement) stems from the fact that different entities are responsible for evaluation (auditing) and transfer (tax) assignments. Moreover, as the provincial (firm's) economic activity is high-dimensional, part of the Bureau of Statistics' (auditor's) job is to simplify the task of transfer (tax) assignment for the central government.

The key novelty in our model is that the allocation rule allows for randomization and that transfers depend on the allocation outcome rather than directly on the report. When an allocation rule is deterministic, whether the transfer depends on the report or the outcome is irrelevant. This is also known as the taxation principle or tariff principle. In real life, we observe tariffs more often than direct revelation mechanisms, because the set of possible type spaces may be hard to describe in reality (Tadelis and Segal, 2005). So simplicity is a prominent advantage of tariffs and our mechanism directly inherits this advantage when the type space is large.

Another motivation for our mechanism is that it could be useful in other mechanism design problems. The standard mechanism with report-contingent transfer specifies a mapping that assigns to each type an allocation and a transfer. Our model, instead, decouples this mapping into two functions, the allocation rule $\pi\colon \Theta \to \Delta(X)$ and the transfer rule $t\colon X \to \mathbb{R}$. This decomposition also appears in the canonical mechanism in Doval and Skreta (2022). In their leading example (see their Section 3.1), the canonical mechanism is a mechanism with outcome-contingent transfers.[4]

[3]The transfer's contingency on evaluations and the randomness in the evaluation/allocation rule also arise from confidentiality concerns. If the transfer were to depend directly on the state, it would reveal too much information about the state, as the transfer is publicly observable.

[4]In their leading example, the disclosure rule maps a report to a distribution over posterior beliefs,

Our paper answers the question of when such transfers exist under the truthful report (IC) constraint.

The taxation principle states that when an allocation rule is deterministic, whether the transfer depends on the report or the outcome is irrelevant. However, if the allocation rule is randomized, we show that it is harder to implement with outcome-contingent transfers (see Observation 1). We say that the allocation rule is *implementable* with outcome-contingent transfers if there exist outcome-contingent transfers such that it is optimal for the agent to report truthfully in each state.

Our main result is a characterization of implementable allocation rules. We collect the agent's valuation of all outcomes in state θ into a $|X|$-dimensional vector $v(\theta)$. For each pair of states (θ, θ') we define the *allocation difference* as the difference in probabilities

$$d\pi(\theta, \theta') = \pi(\theta) - \pi(\theta')$$

and the *valuation loss* as the agent's difference in valuation in the two states:

$$\mathrm{vl}(\theta, \theta') = d\pi(\theta, \theta') \cdot v(\theta).$$

We collect the allocation differences into a $|X| \times |\Theta|^2$ dimensional matrix $D\pi$ consisting columns of $d\pi(\theta, \theta')$, and all valuation losses $\mathrm{vl}(\theta, \theta')$ into a valuation loss vector VL, by ordering pairs of states (θ, θ') in the same order.

Recall that a matrix's positive kernel $\ker_+$ is the intersection of the kernel and the positive orthant. We show that the allocation rule is implementable if and only if VL lies in the dual cone of $\ker_+(D\pi)$ (Theorem 1). Moreover, we offer another geometric characterization. Each pair of distinct states (θ, θ') determines an allocation difference $d\pi(\theta, \theta')$ and a valuation loss $\mathrm{vl}(\theta, \theta')$. This $(d\pi(\theta, \theta'), \mathrm{vl}(\theta, \theta'))$ corresponds to a point in $\mathbb{R}^{|X|} \times \mathbb{R}$. For all such points, we can construct their convex envelope $conv \colon \mathbb{R}^{|X|} \to \mathbb{R}$. We show that the allocation rule is implementable if and only if the convex envelope's intercept $conv(\mathbf{0})$, the value of $conv$ evaluated at $\mathbf{0}$, is weakly positive (Theorem 1).

Moreover, when the allocation rule is implementable with outcome-contingent transfers, we show how transfer payments can be constructed (Proposition 1). In particular, we show that transfers can be recovered from the subgradient of the convex

and the price depends only on the realized posterior belief. Their posterior belief is our allocation outcome and their price is our transfer.

envelope at $d\pi = \mathbf{0}$. When the convex envelope intercept is strictly positive, we show that the allocation rule is strictly implementable (Proposition 2).[5]

In addition, we show that the classic cyclic monotonicity condition of Rochet (1987) is a necessary condition for implementation in our setup as well (Observation 1). However, without further assumptions on the valuation structure and the allocation rule, cyclic monotonicity is not sufficient in general. Yet, we show that cyclic monotonicity is also sufficient when the allocation measures $\{\pi(\theta)\}_{\theta\in\Theta}$ are linearly independent (Proposition 3). Additionally, when there are fewer than exactly four states, cyclic monotonicity is also sufficient (Proposition 5). Furthermore, when $\{\pi(\theta)\}_{\theta\in\Theta}$ are convex dependent, we show that it is without loss to only check whether one candidate transfer can implement the allocation rule (Proposition 4).

1.1 Literature Review

Our paper contributes to the literature on implementation by studying randomized allocation rules with outcome-contingent transfers. The implementation literature has explored when allocation rules can be truthfully implemented by transfer that depends on the *report*; see Roberts (1979); Rochet (1987); McAfee and McMillan (1988); Jehiel et al. (1999); Gui et al. (2004); Saks and Yu (2005); Bikhchandani et al. (2006); Müller et al. (2007); Archer and Kleinberg (2008); Ashlagi et al. (2010); Bergemann et al. (2012); Carroll (2012); Carbajal and Müller (2015, 2017); Kushnir and Lokutsievskiy (2021); Frongillo and Kash (2021).

For single agent settings, Myerson (1981) shows the implementability condition is the subgradient condition in a one-dimensional continuous-type environment. Müller et al. (2007) and Archer and Kleinberg (2008) propose several equivalent conditions. Rochet (1987) studies when an allocation rule is implementable in dominant strategy mechanisms. He shows that the cyclic monotonicity condition is sufficient and necessary for an allocation rule to be implementable. Bergemann et al. (2012) analyze this implementability problem in Bayesian incentive-compatible mechanisms.

In quasilinear environments with a complete domain, Roberts (1979) shows that a positive association of differences is necessary and sufficient for dominant-strategy incentive compatibility. In addition, he derives another characterization in terms of

[5] We say that the allocation rule is *strictly implementable* with outcome-contingent transfers if there exist outcome-contingent transfers such that it is strictly optimal for the agent to report truthfully in each state.

affine maximizers. For a selection of restricted domains, Bikhchandani et al. (2006) characterize dominant-strategy incentive compatibility by weak (cyclic) monotonicity. Gui et al. (2004) notice that this result holds for the unrestricted domain and for every cube. Saks and Yu (2005) extend this result to any convex multi-dimensional type space. Ashlagi et al. (2010) shows that if the closure of a domain is not convex, then there exists a finite-valued monotone allocation rule that is not implementable. Several more recent works (Kushnir and Lokutsievskiy, 2021; Carbajal and Müller, 2015, 2017) also identify some conditions under which weak monotonicity (2-cycle monotonicity) is sufficient to implement the allocation rule. Frongillo and Kash (2021) provide a unified framework nesting mechanisms and scoring rules and characterize scoring rules for non-convex sets of distributions.

From a modeling perspective, our work is closely related to the literature of Bayesian persuasion. Our randomized allocation rule can equivalently be seen as a Blackwell experiment.[6] Perez-Richet and Skreta (2022) study the receiver-optimal Blackwell experiment when the sender can falsify the state of the world as the input of the experiment at some cost. Lin and Liu (2024) study when a Blackwell experiment is credible, i.e., when the sender cannot profitably deviate to another experiment while fixing the marginal distribution of realizations. Their credibility also boils down to a cyclic monotonicity condition. Yet, their cyclic monotonicity condition is ex-post rather than ex-ante.

2 The Model

Primitives. We are given a finite set Θ of states. (Our main result Theorem 1 has two sufficient and necessary conditions. The cone condition only applies to finite state spaces. For infinite state space, the envelope condition still holds.) An agent observes the state and sends a report to a predetermined allocation rule. An *allocation rule* π maps a reported state to a distribution over a finite set of outcomes $X = \{x_1, \ldots, x_n\}$, i.e., $\pi \colon \Theta \to \Delta(X)$. Thus each $\pi(\theta)$ is a probability measure over the outcome space

[6] Nguyen and Tan (2021) study a model of Bayesian persuasion where the sender does not observe the underlying state, commits to a Blackwell experiment, and privately observes the experiment realization. The sender can misrepresent the experiment's realization with some lying cost. The cost depends on both the experiment realization and the message sent.

X. We denote by $\pi(x|\theta)$ the probability assigned to x by the measure $\pi(\theta)$.

$$\pi(\theta) = (\pi(x_1|\theta), \pi(x_2|\theta), \cdots, \pi(x_n|\theta))^\top.$$

The allocation rule can be seen as outside of the planner's influence, as in our motivating examples, or can be interpreted as the planner's objective. The agent's valuation for outcome x in state θ is equal to $v(\theta, x)$. We write $v(\theta)$ as the $|X|$-dimensional vector consisting of entries $v(\theta, x)$ for all $x \in X$,

$$v(\theta) = (v(\theta, x_1), v(\theta, x_2), \cdots, v(\theta, x_n))^\top.$$

Transfer Design. The agent's total payoff is linear in the valuation and the transfer. The transfer depends only on the outcome and we let $t(x)$ denote the transfer to the agent given outcome x. We use T to denote the n-dimensional transfer vector consisting of entries $t(x)$ for all $x \in X$,

$$T = (t(x_1), t(x_2), \cdots, t(x_n))^\top.$$

We say that the allocation rule is *implementable with outcome-contingent transfers* if there exists a vector T such that for all θ, $\theta' \in \Theta$,

$$\pi(\theta) \cdot (v(\theta) + T) \geq \pi(\theta') \cdot (v(\theta) + T).$$

Compared to the standard implementation with report-contingent transfers, our notion has the additional requirement that the transfer is linear in the allocation rule π. That is, the report-contingent transfer takes the form of $\pi(\theta) \cdot T$.

For any transfer vector T that satisfies this incentive compatibility constraint, any translation $T + (c, c, \cdots, c)$ also satisfies the condition. Thus, we are free to impose an ex-ante budget balance condition. For example, suppose we are given a prior distribution over the states. Given truthful reports, the allocation rule induces a distribution over outcomes. Then we can impose an ex-ante budget balance condition, i.e., $E_x t(x) = 0$ where E_x is the expectation with respect to the random outcome x.

Problem Reformulation. Next, we reformulate the implementation problem into the following geometric form. Given the set $\mathbf{\Pi} = \{\pi(\theta)|\theta \in \Theta\}$ in $\mathbb{R}^n$, we associate to

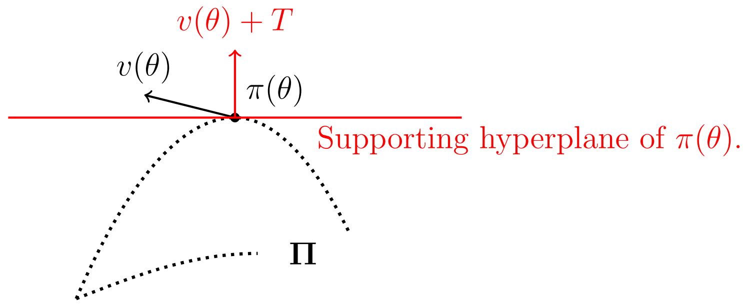

Figure 1: The Supporting Hyperplane

each vector $\pi(\theta)$ the vector $v(\theta) \in \mathbb{R}^n$. We ask if there exists a vector $T \in \mathbb{R}^n$ such that for all $\pi(\theta)$, $T + v(\theta)$ is the outer normal of a supporting hyperplane of the set $\mathbf{\Pi}$ at $\pi(\theta)$. That is, for all θ, $\theta' \in \Theta$,

$$[\pi(\theta) - \pi(\theta')] \cdot (v(\theta) + T) \geq 0.$$

Informally, we want a common adjustment vector T such that for all the points in $\mathbf{\Pi}$, the new vector $v + T$ is the outer normal of a supporting hyperplane of the set $\mathbf{\Pi}$ (see Figure 1).

3 Main Results

We first provide an example where the allocation rule is implementable with standard report-contingent transfers but not with outcome-contingent ones.

Example 1. There are four possible types and two outcomes: $\Theta = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $X = \{x_1, x_2\}$. The allocation rule is

$$\pi(\theta) = (\theta, 1 - \theta)^\top.$$

The valuation vector is

$$v(\theta) = (\theta, 0)^\top.$$

Note that report-contingent transfer $\tilde{t}(\theta) = -\frac{1}{2}\theta^2$ can implement the allocation rule.

We show that π is not implementable with outcome-contingent transfers and the agent always has an incentive to misreport. For the agent with type θ, the payoff difference between truthful reporting and misreporting θ' is

$$(\theta - \theta')(\theta + t(x_1) - t(x_2)).$$

If $t(x_1)-t(x_2)+\theta > 0$, the agent would prefer to report $\theta' = 1$. If $t(x_1)-t(x_2)+\theta < 0$, the agent would prefer to report $\theta' = 0$. The agent with type θ that is not 0 or 1 would truthfully report if and only if $\theta = t(x_2) - t(x_1)$. Thus, any transfer that elicits type 1/3 cannot elicit type 2/3. This example shows that our implementation notion is stronger than the standard one.

We call

$$d\pi(\theta, \theta') = \pi(\theta) - \pi(\theta')$$

the *allocation difference* between state θ and θ'. We define the *valuation loss* for state θ when inputting θ' to be the agent's expected loss on the valuation

$$\mathrm{vl}(\theta, \theta') = d\pi(\theta, \theta') \cdot v(\theta).$$

We can calculate the agent's expected deviation loss in state θ when reporting θ' as $\mathrm{vl}(\theta, \theta') + d\pi(\theta, \theta') \cdot T$. Then the incentive compatibility constraint can be written as for all θ, $\theta' \in \Theta$,

$$\mathrm{vl}(\theta, \theta') + d\pi(\theta, \theta') \cdot T \geq 0.$$

Let $\mathrm{VL} \in R^{|\Theta| \times |\Theta|}$ denote the vector consisting entries $\mathrm{vl}(\theta, \theta')$ by numerating all (θ, θ') pairs. We let $D\pi$ denote a $|X| \times |\Theta|^2$ dimensional matrix consisting columns $d\pi(\theta, \theta')$ with (θ, θ') arranged in the same order as VL. We define the *positive kernel* of $D\pi$ to be

$$\ker_+(D\pi) = \left\{ \lambda \in \mathbb{R}_+^{|\Theta| \times |\Theta|} : \sum_{\theta, \theta' \in \Theta} \lambda_{\theta\theta'} d\pi(\theta, \theta') = 0 \right\}.$$

This set is non-empty as we can set $\lambda_{\theta_1\theta_2} = \lambda_{\theta_2\theta_1} = 1$ and all other entries to be zero. Since the positive kernel is the intersection between the kernel of $D\pi$ (a linear subspace) and the nonnegative orthant, it is a finitely generated convex cone. Its dual cone is given by

$$[\ker_+(D\pi)]^* = \{ y \in \mathbb{R}^{|\Theta| \times |\Theta|} \, | y \cdot \lambda \geq 0, \forall \lambda \in \ker_+(D\pi) \}.$$

We append the valuation loss to the allocation difference vector to get a new set of vectors

$$D = \{(d\pi(\theta, \theta'), \text{vl}(\theta, \theta')) | \theta \neq \theta' \in \Theta\},$$

which we call the *difference set*. We let $conv(D)$ denote the *convex envelope* of the set D

$$conv(D)(\cdot) = \sup\{g(\cdot) | g \colon \mathbb{R}^n \to \mathbb{R} \text{ is convex and } g(d\pi(\theta, \theta')) \leq \text{vl}(\theta, \theta'), \forall \theta \neq \theta' \in \Theta\}.$$

We call $conv(D)(\mathbf{0})$ the *convex envelope intercept*.

Now we are ready to characterize the implementation condition.

Theorem 1. *The following are equivalent.*

1. *The allocation rule is implementable with outcome-contingent transfers.*
2. $\text{VL} \in [\ker_+(D\pi)]^*$.
3. *The convex envelope intercept is weakly positive.*

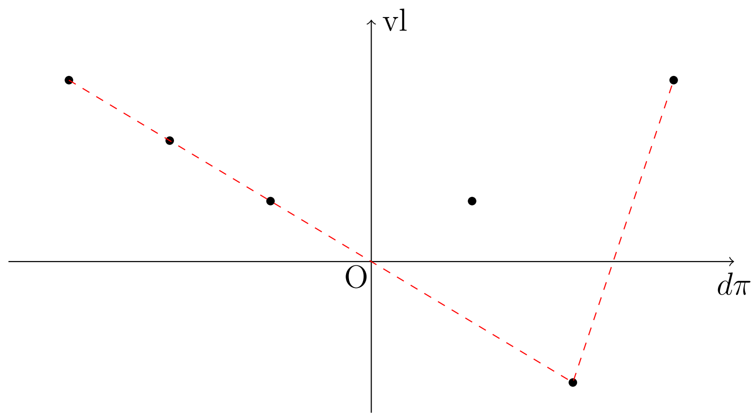

Figure 2: An Illustration of Convex Envelope

We provide a geometric example to use the last condition in Theorem 1 to check implementation. Suppose there are three states $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and two outcomes. The allocation rule is $\pi(\theta_1) = (1, 0)^T, \pi(\theta_2) = (\frac{2}{3}, \frac{1}{3})^T$ and $\pi(\theta_3) = (0, 1)^T$. The agent's valuations are $v(\theta_1) = (1, 0)^T, v(\theta_2) = (1, 2)^T, v(\theta_3) = (0, 1)^T$. Then we construct the difference set

$$D = \{((\frac{1}{3}, -\frac{1}{3}), \frac{1}{3}), ((-\frac{1}{3}, \frac{1}{3}), \frac{1}{3}), ((1, -1), 1), ((-1, 1), 1), ((\frac{2}{3}, -\frac{2}{3}), -\frac{2}{3}), ((-\frac{2}{3}, \frac{2}{3}), \frac{2}{3})\}.$$

Since the entries of $d\pi(\theta, \theta')$ sum up to zero, we can identify each $d\pi$ with an element in $\mathbb{R}$ and plot them in Figure 2. The red dashed line is the convex envelope of D and the value of the convex envelope evaluated at $\mathbf{0}$ is 0. So by Theorem 1, the allocation rule is implementable with outcome-contingent transfers.

We illustrate the intuition for the necessity of the convex envelope intercept condition. Suppose $\Theta = \{\theta_1, \theta_2, \cdots\}$ and $\mathrm{vl}(\theta_1, \theta_2) + \mathrm{vl}(\theta_2, \theta_1) < 0$ (see the left panel of Figure 3). Then, $\mathrm{vl}(\theta_1, \theta_2) + \mathrm{vl}(\theta_2, \theta_1) < 0$ implies $conv(D)(\mathbf{0}) < 0$, as $d\pi(\theta_1, \theta_2) = -d\pi(\theta_2, \theta_1)$. Similar to Example 1, any transfer T that elicits θ_1 cannot elicit θ_2. To elicit θ_1, we must have $d\pi(\theta_1, \theta_2) \cdot T + \mathrm{vl}(\theta_1, \theta_2) \geq 0$. As $d\pi(\theta_1, \theta_2) = -d\pi(\theta_2, \theta_1)$, the transfer T must have the opposite effect on type θ_2. That is, whenever we use some transfer to ensure $d\pi(\theta_1, \theta_2) \cdot T + \mathrm{vl}(\theta_1, \theta_2) \geq 0$, this leads to $d\pi(\theta_2, \theta_1) \cdot T + \mathrm{vl}(\theta_2, \theta_1) < 0$, as the intercept is preserved (see the right panel of Figure 3). In fact, the condition $\mathrm{vl}(\theta_1, \theta_2) + \mathrm{vl}(\theta_2, \theta_1) \geq 0$ is exactly the weak monotonicity and so it must hold for implementation. But the opposing effect around the intercept is driving the necessity of $conv(D)(\mathbf{0}) \geq 0$. The intuition carries over in general such that if $conv(D)(\mathbf{0}) < 0$, for all transfer T, at least one pair (θ_i, θ_j) has $d\pi(\theta_i, \theta_j) \cdot T + \mathrm{vl}(\theta_i, \theta_j) < 0$.

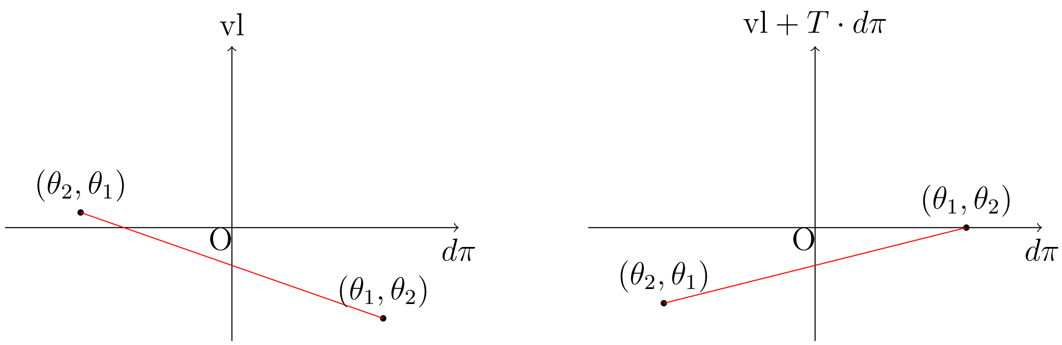

Figure 3: The Effect of Transfer

Next, we prove the necessity of the second statement, i.e., $1 \Rightarrow 2$. If the allocation rule is implementable with outcome-contingent transfers, there exists T such that for all $\theta, \theta' \in \Theta$

$$\mathrm{vl}(\theta, \theta') + d\pi(\theta, \theta') \cdot T \geq 0.$$

For any $\lambda_{\theta\theta'} > 0$, we have

$$\lambda_{\theta\theta'}(\mathrm{vl}(\theta, \theta') + d\pi(\theta, \theta') \cdot T) \geq 0.$$

For any $\lambda \in \ker_+(D\pi)$, summing over all θ, θ', the second term is zero. What left is

$$\sum_{\theta,\theta'} \lambda_{\theta\theta'} \mathrm{vl}(\theta, \theta') \geq 0.$$

Since this holds for all $\lambda \in \ker_+(D\pi)$, we have $\mathrm{VL} \in [\ker_+(D\pi)]^*$.

The equivalence between statements 2 and 3 follows by a property of the convex envelope (see Boyd and Vandenberghe (2004) p.119)

$$conv(D)(\mathbf{z}) = \inf \left\{ \sum_{\theta,\theta' \in \Theta} \lambda_{\theta\theta'} \mathrm{vl}(\theta, \theta') \middle| \sum_{\theta \neq \theta' \in \Theta} \lambda_{\theta\theta'} = 1, \lambda_{\theta\theta'} \geq 0, \sum_{\theta,\theta' \in \Theta} \lambda_{\theta\theta'} d\pi(\theta, \theta') = \mathbf{z} \right\},$$

$$conv(D)(\mathbf{0}) = \inf \left\{ \lambda \cdot \mathrm{VL} \middle| \sum_{\theta \neq \theta' \in \Theta} \lambda_{\theta\theta'} = 1, \lambda \in \ker_+(D\pi) \right\}.$$

Then $conv(D)(\mathbf{0}) \geq 0$ is equivalent to $\lambda \cdot \mathrm{VL} \geq 0$ for all $\lambda \in \ker_+(D\pi)$, which is $\mathrm{VL} \in [\ker_+(D\pi)]^*$

The sufficiency condition states that as long as the convex envelope intercept is weakly positive, the allocation rule is implementable with outcome-contingent transfers. Now we construct a transfer T such that for all $\theta, \theta' \in \Theta$, $\mathrm{vl}(\theta, \theta') + d\pi(\theta, \theta') \cdot T \geq 0$. Suppose $conv(D)(\mathbf{0}) \geq 0$. Let T be the negative of any subgradient of $conv(D)(\cdot)$ at $d\pi = \mathbf{0}$, i.e.,

$$-T \in \partial conv(D)(\mathbf{0})$$

where $\partial conv(D)(\mathbf{0})$ denotes the subdifferential of $conv(D)(\cdot)$ at $\mathbf{0}$. By the definition of the convex envelope and subgradient, for all $\theta, \theta' \in \Theta$,

$$\mathrm{vl}(\theta, \theta') \geq conv(D)(d\pi(\theta, \theta')) \geq conv(D)(\mathbf{0}) - T \cdot d\pi(\theta, \theta').$$

$$\mathrm{vl}(\theta, \theta') + d\pi(\theta, \theta') \cdot T \geq conv(D)(\mathbf{0}) \geq 0.$$

Geometrically, we rotate the difference set D around $(\mathbf{0}, conv(D)(\mathbf{0}))$ such that the convex envelope is above the $\mathrm{vl} = 0$ plane while preserving the intercept with the vl-axis. In the example in Figure 2, all deviation losses will be positive after the rotation, as shown in Figure 4.

We summarize the construction of the transfer.

Proposition 1. *If the allocation rule is implementable with outcome-contingent trans-*

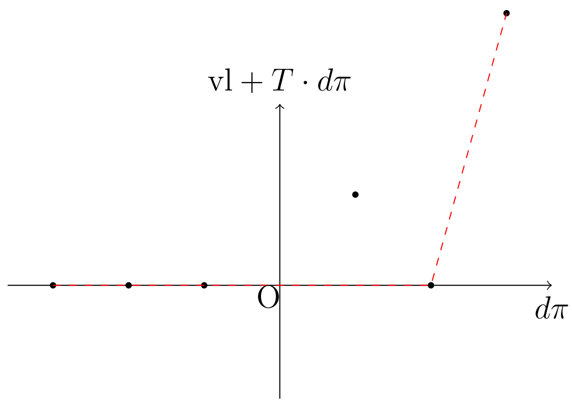

Figure 4: The Deviation Loss with Transfer

fers, any $T \in -\partial conv(D)(\mathbf{0})$ *can implement the allocation rule.*

The convex envelope intercept provides a measure of robustness of the implementation. Given the transfer identified above, the deviation loss is always above $conv(D)(\mathbf{0})$. (This is reminiscent of the definition of ϵ-Nash equilibrium.) Given this observation, we can characterize when an allocation rule is strictly implementable, i.e., the incentive to report truthfully is strict. We say that the allocation rule is *strictly implementable* if there exists a transfer T such that for all $\theta \neq \theta' \in \Theta$,

$$\pi(\theta) \cdot (v(\theta) + T) > \pi(\theta') \cdot (v(\theta) + T).$$

Proposition 2. *If the convex envelope intercept is strictly positive, the allocation rule is strictly implementable.*

Similar to the definition of ϵ-Nash equilibrium, we can also adopt a weaker condition on implementation. For any ϵ, we say that an allocation rule is *ϵ-implementable with outcome-contingent transfers* if there exists a transfer T such that the total gain from deviation is always less than ϵ, i.e., for all θ, $\theta' \in \Theta$,

$$\pi(\theta') \cdot (v(\theta) + T) - \pi(\theta) \cdot (v(\theta) + T) \leq \epsilon.$$

Corollary 1. *The allocation rule is* $-conv(D)(\mathbf{0})$*-implementable.*

4 Discussions

Rochet (1987) studies the implementability condition with report-contingent transfer and shows that the cyclic monotonicity is sufficient and necessary. Formally, cyclic monotonicity is equivalent to the existence of a function $\tilde{t} : \Theta \to \mathbb{R}$ such that for all θ, θ',

$$\pi(\theta) \cdot v(\theta) + \tilde{t}(\theta) \geq \pi(\theta') \cdot v(\theta) + \tilde{t}(\theta').$$

Note the difference between report-contingent transfer and outcome-contingent transfer. Our implementation additionally requires that $\tilde{t}$ is linear in the allocation rule. If an allocation rule is implementable with outcome-contingent transfers, then the function $\tilde{t}$ must exist: $\tilde{t}(\theta) = \pi(\theta) \cdot T$. We thus obtain a necessary condition.

Observation 1. The allocation rule is implementable with outcome-contingent transfers only if $\mathrm{vl}(\cdot, \cdot)$ satisfies cyclic monotonicity, i.e., for all $\theta_1, \cdots, \theta_k \in \Theta$,

$$\mathrm{vl}(\theta_1, \theta_2) + \mathrm{vl}(\theta_2, \theta_3) + \cdots + \mathrm{vl}(\theta_k, \theta_1) \geq 0.$$

Yet, the condition of cyclic monotonicity does not guarantee the existence of outcome-contingent transfers. In Example 1, a report-contingent transfer can implement the allocation rule. Thus, cyclic monotonicity holds. However, no outcome-contingent transfer can implement the allocation rule. Hence, our condition in Theorem 1 is stronger than cyclic monotonicity.

4.1 Special Cases

Our model imposes no assumptions on the allocation rule, the state space, or the valuation structure. Next, we shall investigate some special cases where we impose more structure on each of these model primitives. We first show that when $\{\pi(\theta)\}_{\theta \in \Theta}$ are linearly independent, cyclic monotonicity is also sufficient.

Proposition 3. *When $\{\pi(\theta)\}_{\theta \in \Theta}$ are linearly independent, the allocation rule is implementable with outcome-contingent transfers if and only if* $\mathrm{vl}(\cdot, \cdot)$ *satisfies cyclic monotonicity.*

Proof of Proposition 3. Observation 1 shows the necessity. We only need to show the sufficiency. By our previous discussion, cyclic monotonicity already ensures the existence of $\tilde{t}$. We only need to show that there exists a transfer $T \in \mathbb{R}^n$ such that for all $\theta \in \Theta$,

$$\pi(\theta) \cdot T = \tilde{t}(\theta).$$

We rewrite it in matrix form. We define $\mathbf{\Pi}$ be the $|\Theta| \times n$ matrix representing $\pi : \theta \to \Delta(X)$ and $\tilde{T}$ be the $|\Theta|$-dimension column vector representing $\tilde{t}(\cdot) : \Theta \to \mathbb{R}$. So the matrix form of the above linear system is

$$\mathbf{\Pi} T = \tilde{T}. \tag{1}$$

The vector $\tilde{T}$ lies in the span of column vectors of $\mathbf{\Pi}$. Thus, there exists a T satisfying the above matrix equation if and only if $rank(\mathbf{\Pi}) = rank(\mathbf{\Pi}, \tilde{T})$ where $\mathbf{\Pi}, \tilde{T}$ represents the augmented matrix. Since $\pi(\theta)$ is linearly independent, $rank(\mathbf{\Pi}) = |\Theta| = rank(\mathbf{\Pi}, \tilde{T})$. □

This proposition highlights the difference between allocation rules that are implementable with outcome-contingent transfers versus report-contingent transfers. If the cardinality of the outcomes is larger than the cardinality of states, then we have more flexibility in setting $t(x)$ to induce truthful reports. That is, generically, a matrix Π such that $|X| \geq |\Theta|$ guarantees the existence of a solution to Equation (1). Conversely, when $|X| < |\Theta|$, it is more likely that no solution exists. This insight sheds light on the motivating examples. How far are the allocation rules that are implementable with outcome-contingent transfers compared to the ones with standard transfers? The answer largely lies in the granularity of outcomes versus states. In the political transfer example, as the state is high-dimensional, it is very hard to implement with outcome-contingent transfers if the evaluations are finite. However, the two implementation notions are closer given a larger evaluation set.

Next, we show that in some cases, it is easy to check whether an allocation rule is implementable. Consider the support problem in Figure 5. The outcome-contingent transfer rule is the common vector adjustment in the vector field that makes $v(\theta) + T$ (the outer normal of) a supporting hyperplane. Consider the point $\pi(\theta_4)$ in Figure 5. As $\pi(\theta_4)$ is in the convex hull of $\{\pi(\theta) | \theta \in \Theta\}$, no hyperplane can support $\{\pi(\theta) | \theta \in \Theta\}$. Therefore, the outer normal $v(\theta_4) + T$ must be zero and $T = -v(\theta_4)$.[7]

[7] Formally, $v(\theta_4) + T$ can be non-zero but must be orthogonal to the affine hull of $\{\pi(\theta) | \theta \in \Theta\}$. But then it is without loss to consider only $T = -v(\theta_4)$.

Consequently, this is the only candidate transfer that we need to check. This insight carries over in general.

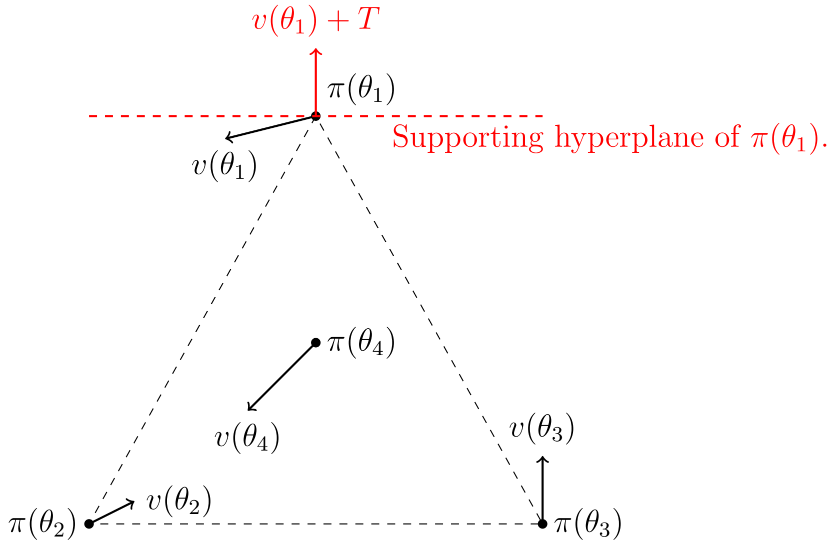

Figure 5: The Support Problem

Proposition 4. *Suppose that some $\pi(\theta_i)$ is in the interior of the convex hull of $\{\pi(\theta)|\theta \in \Theta\}$. The allocation rule π is implementable with outcome-contingent transfers if and only if $T = -v(\theta_i)$ implements π.*

We can apply this result to our Example 1. As $\pi(\theta_2)$ is in the interior of the convex hull of $\{\pi(\theta)|\theta \in \Theta\}$, it is without loss to consider only the transfer $T = -v(\theta_2)$. But this transfer cannot elicit θ_3 to report truthfully. Thus, the allocation rule is not implementable.

Note that this proposition has no bite if the points $\{\pi(\theta)|\theta \in \Theta\}$ are in a convex position (also known as convex independent). On the other hand, when $\{\pi(\theta)|\theta \in \Theta\}$ are convex dependent, implementation is generally difficult. Even when such an allocation can be implemented, the transfer rule is very restricted. In Appendix B, we consider the planner's design problem where he optimizes over allocation-transfer rules. We show that for a general objective function, it is without loss to restrict attention to convex independent allocation rules.

In addition, we show that when the state space is small, cyclic monotonicity is also sufficient.

Proposition 5. *When $|\Theta| \leq 3$, the allocation rule is implementable with outcome-contingent transfers if and only if* $\mathrm{vl}(\cdot,\cdot)$ *satisfies cyclic monotonicity.*

By Proposition 3, the conclusion follows when $\pi(\theta)$ are linearly independent. Now suppose $\pi(\theta)$ are linearly dependent. When $|\Theta| = 2$, linear dependence of $\pi(\theta)$ implies $\pi(\theta_1) = \pi(\theta_2)$ and the conclusion holds trivially. When $|\Theta| = 3$, the linear dependence of $\pi(\theta)$ is equivalent to convex dependence. Suppose that $\pi(\theta_1)$ is a convex combination of $\pi(\theta_2)$ and $\pi(\theta_3)$. By Proposition 4, it is without loss to take $T = -v(\theta_1)$. Given this transfer, all the incentive compatibility constraints reduce to weak monotonicity.[8] Thus, cyclic monotonicity is also sufficient.

The argument above fails catastrophically for $|\Theta| \geq 4$. First, when there are more than three states, linear dependence does not imply convex dependence. Second, even if convex dependence of $\{\pi(\theta)|\theta \in \Theta\}$ holds, we can no longer reduce all IC constraints to weak monotonicity. This occurs in Example 1, where $|\Theta| = 4$ and cyclic monotonicity holds. Thus, when $|\Theta| \geq 4$, cyclic monotonicity may no longer be sufficient.

In the incentive compatibility constraint, we take expectation of the random outcome. Thus, we can view the implementation with outcome-contingent transfers as an interim condition. A more demanding notion can require the allocation rule to be ex-post implementable with outcome-contingent transfers, i.e., if there exists a transfer t such that for all θ and $x \in \mathbf{supp}\{\pi(\cdot|\theta)\}$,

$$v(\theta, x) + t(x) \geq v(\theta, x') + t(x'), \forall x' \in X.$$

It turns out that this ex-post implementability is equivalent to the following sufficient condition on the valuation structure.[9]

[8]To see this, let $\pi(\theta_1) = \lambda\pi(\theta_2) + (1-\lambda)\pi(\theta_3)$. Given $T = -v(\theta_1)$, the incentive compatibility requires that for all $i \neq j \in \{1, 2, 3\}$,

$$\pi(\theta_i) \cdot (v(\theta_i) - v(\theta_1)) \geq \pi(\theta_j) \cdot (v(\theta_i) - v(\theta_1)).$$

When $i = 1$, the inequality holds trivially. When $j = 1$, the inequality reduces to weak monotonicity between θ_i and θ_1. When $i, j \in \{2, 3\}$, replacing $\pi(\theta_j)$ with

$$\frac{\pi(\theta_1) - \lambda_i \pi(\theta_i)}{1 - \lambda_i}$$

where $\lambda_i = \lambda$ if $i = 2$ and $\lambda_i = 1 - \lambda$ otherwise, the inequality also reduces to weak monotonicity between θ_i and θ_1.

[9]We thank one anonymous referee for providing this result.

Proposition 6. *The allocation rule is implementable with outcome-contingent transfers if for any sequence of* $(\theta_1, x_1), \cdots, (\theta_m, x_m), (\theta_{m+1}, x_{m+1}) = (\theta_1, x_1)$ *where* $x_i \in \boldsymbol{supp}\{\pi(\cdot|\theta_i)\}$,

$$\sum_{i=1}^{m} v(\theta_i, x_i) \geq \sum_{i=1}^{m} v(\theta_i, x_{i+1}).$$

Another important case is when the agent's preference is separable in the state and outcome. We say that the agent's preference is *additively separable* if there exists v_1 and v_2 such that

$$v(\theta, x) = v_1(\theta) + v_2(x).$$

It includes the case where the agent's preference is state-independent. State-independent preference, or transparent motive, has been widely studied in the communication and persuasion literature (see, for example, Chakraborty and Harbaugh, 2010; Lipnowski and Ravid, 2020; Lipnowski et al., 2022). When the agent's preference is additively separable, the transfer $t(x) = -v_2(x)$ can implement all allocation rules. Moreover, the converse is true as well, i.e., this is the only preference where all allocation rules are implementable with outcome-contingent transfers.

5 Conclusion

We study whether we can implement a randomized allocation rule with outcome-contingent transfers. For this implementation, we characterize sufficient and necessary conditions. One natural extension is to study a principal's revenue maximization problem when the agent reports through a noisy signal, which can be viewed as our allocation rule. The principal allocates one indivisible item conditional on the message generated by the signal. Given the agent's IR constraint, the principal designs transfers to maximize revenue. We leave this question to future research.

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A Omitted Proofs

Proof of Theorem 1. 1 $\Rightarrow$ 2. If the allocation rule is implementable with outcome-contingent transfers, there exists T such that for all $\theta, \theta' \in \Theta$

$$\text{vl}(\theta, \theta') + d\pi(\theta, \theta') \cdot T \geq 0.$$

For any $\lambda_{\theta\theta'} > 0$, we have

$$\lambda_{\theta\theta'}(\text{vl}(\theta, \theta') + d\pi(\theta, \theta') \cdot T) \geq 0.$$

For any $\lambda \in \ker_+(D\pi)$, summing over all θ, θ',

$$\sum_{\theta,\theta'} \lambda_{\theta\theta'} \text{vl}(\theta, \theta') \geq 0$$

we have $\text{VL} \in [\ker_+(D\pi)]^*$.

2 $\Rightarrow$ 3. If $\text{VL} \in [\ker_+(D\pi)]^*$, the optimal value of following linear programming problem is zero.

$$\begin{aligned} \min \sum_{\theta,\theta' \in \Theta} & \lambda_{\theta\theta'} \text{vl}(\theta, \theta') \\ s.t. \sum_{\theta,\theta' \in \Theta} & \lambda_{\theta\theta'} d\pi(\theta, \theta') = 0, \\ & \lambda_{\theta\theta'} \geq 0. \end{aligned} \tag{2}$$

Then, for any $\lambda_{\theta\theta'} \geq 0$ satisfies $\sum_{\theta,\theta' \in \Theta} \lambda_{\theta\theta'} d\pi(\theta, \theta') = 0$ and $\sum_{\theta \neq \theta' \in \Theta} \lambda_{\theta\theta'} = 1$, we have $\sum_{\theta,\theta' \in \Theta} \lambda_{\theta\theta'} \text{vl}(\theta, \theta') \geq 0$.

By the definition of convex envelope,[10]

$$conv(D)(\mathbf{z}) = \inf\{\sum_{\theta,\theta' \in \Theta} \lambda_{\theta\theta'} \text{vl}(\theta, \theta') \big| \sum_{\theta \neq \theta' \in \Theta} \lambda_{\theta\theta'} = 1, \lambda_{\theta\theta'} \geq 0, \sum_{\theta,\theta' \in \Theta} \lambda_{\theta\theta'} d\pi(\theta, \theta') = \mathbf{z}\}.$$

Thus we get $conv(D)(\mathbf{0}) \geq 0$.

3 $\Rightarrow$ 1. Since $conv(D)(0) \geq 0$, the convex hull of set D, $conhull(D)$, and the convex set $\{(\mathbf{0}, l) \big| l < 0, \mathbf{0} \in \mathbb{R}^n\}$ have no intersection. By Separating Hyperplane

[10] Here we adopt the convention that $\inf \emptyset = +\infty$.

Theorem, there exists a non-zero vector $(\bar{T}, \alpha)$ where $\bar{T} \in \mathbb{R}^n, \alpha \geq 0$ such that for any $(d\pi(\theta, \theta'), \mathrm{vl}(\theta, \theta')) \in D$ and $l < 0$ we have

$$d\pi(\theta, \theta') \cdot \bar{T} + \alpha \mathrm{vl}(\theta, \theta') > \alpha l \tag{3}$$

If $\alpha = 0$, we get $d\pi(\theta, \theta') \cdot \bar{T} > 0$ and $d\pi(\theta', \theta) \cdot \bar{T} > 0$. But $d\pi(\theta, \theta') + d\pi(\theta', \theta) = 0$, a contradiction. Then it must be that $\alpha > 0$. Set $T = \frac{\bar{T}}{\alpha}$, then by (3)

$$d\pi(\theta, \theta') \cdot T + \mathrm{vl}(\theta, \theta') > l.$$

As $l < 0$, take the supremum of l,

$$d\pi(\theta, \theta') \cdot T + \mathrm{vl}(\theta, \theta') \geq 0.$$

This implies that T is the transfer that implements the allocation rule. □

Proof of Proposition 4. The "if" part is obvious and we prove the "only if" part. Suppose that the allocation rule $\{\pi(\theta)\}$ is implementable. We assume that the transfer T' implements this allocation rule. Since $\pi(\theta_i)$ is in the interior of the convex hull of $\{\pi(\theta) | \theta \in \Theta\}$, there exists $\lambda(\theta) > 0$ such that $\sum_{\theta \neq \theta_i} \lambda(\theta) = 1$ and $\pi(\theta_i) = \sum_{\theta \neq \theta_i} \lambda(\theta)\pi(\theta)$.

By the incentive-compatible constraint, we have that for any $\theta \neq \theta_i$,

$$\lambda(\theta)\pi(\theta_i) \cdot (v(\theta_i) + T') \geq \lambda(\theta)\pi(\theta) \cdot (v(\theta_i) + T').$$

Sum them up, we get

$$\pi(\theta_i) \cdot (v(\theta_i) + T') \geq \pi(\theta_i) \cdot (v(\theta_i) + T')$$

Consequently, all above inequalities must be equalities

$$\pi(\theta_i) \cdot (v(\theta_i) + T') = \pi(\theta) \cdot (v(\theta_i) + T')$$

for all $\theta \neq \theta_i$.

Next, we verify that $T = -v(\theta_i)$ also implements the allocation rule. For any

$\theta, \theta' \in \Theta$,

$$\begin{aligned}\pi(\theta) \cdot (v(\theta) + T) &= \pi(\theta) \cdot (v(\theta) + T') - \pi(\theta) \cdot (v(\theta_i) + T') \\ &\geq \pi(\theta') \cdot (v(\theta) + T') - \pi(\theta_i) \cdot (v(\theta_i) + T') \\ &= \pi(\theta') \cdot (v(\theta) + T') - \pi(\theta') \cdot (v(\theta_i) + T') \\ &= \pi(\theta') \cdot (v(\theta) + T).\end{aligned}$$

Thus $T = -v(\theta_i)$ implements the allocation rule $\{\pi(\theta)\}_{\theta\in\Theta}$. □

Proof of Proposition 5. Observation 1 shows the necessity. We only need to show sufficiency. When $\{\pi(\theta)\}_{\theta\in\Theta}$ are linearly independent, the conclusion holds by Proposition 3. Now suppose $\{\pi(\theta)\}_{\theta\in\Theta}$ are linearly dependent.

When $|\Theta| = 1$, the problem is trivial. When $\Theta = \{\theta_1, \theta_2\}$, the only linearly dependent case is $\pi(\theta_1) = \pi(\theta_2)$. It trivially satisfies the cyclic monotonicity condition and the allocation rule is implementable.

Now suppose $\Theta = \{\theta_1, \theta_2, \theta_3\}$. The result trivially holds when $\pi(\theta_1) = \pi(\theta_2) = \pi(\theta_3)$. For the other cases, there is a unique $t \in [0, 1]$ such that

$$\pi(\theta_3) = t\pi(\theta_1) + (1 - t)\pi(\theta_2) \tag{4}$$

and $\pi(\theta_1) \neq \pi(\theta_2)$. This holds without loss of generality, since $\{\pi(\theta)\}_{\theta\in\Theta}$ are linearly dependent and $\pi(\theta) \geq \mathbf{0}$. Consequently, the dimension of $\ker_+(D\pi)$ is 1. For any $\lambda \in \ker_+(D\pi)$, the coefficient of $\pi(\theta_i)$ in $\sum_{\theta,\theta'\in\Theta} \lambda_{\theta\theta'} d\pi(\theta, \theta')$ is

$$\sum_{j\neq i} (\lambda_{\theta_i\theta_j} - \lambda_{\theta_j\theta_i}).$$

Since the dimension of the kernel space is 1, by (4), there exists a real number K such that

$$\begin{aligned}\sum_{j\neq 1} (\lambda_{\theta_1\theta_j} - \lambda_{\theta_j\theta_1}) &= Kt \\ \sum_{j\neq 2} (\lambda_{\theta_2\theta_j} - \lambda_{\theta_j\theta_2}) &= K(1 - t) \\ \sum_{j\neq 3} (\lambda_{\theta_3\theta_j} - \lambda_{\theta_j\theta_3}) &= -K\end{aligned} \tag{5}$$

Take any $\lambda \in \ker_+(D\pi)$. If there exists a cycle $(s_1, s_2, \cdots, s_k) \subseteq \Theta$ such that $\lambda_{s_1 s_2} \times \cdots \times \lambda_{s_\kappa s_1} \neq 0$. Then let

$$y = \min\{\lambda_{s_1 s_2}, \cdots, \lambda_{s_\kappa s_1}\} > 0,$$

and update the values of $\lambda_{s_1 s_2}, \lambda_{s_2 s_3}, \cdots, \lambda_{s_\kappa s_1}$ as the following:

$$\begin{aligned} \lambda_{s_1 s_2} &\leftarrow \lambda_{s_1 s_2} - y \\ &\cdots \\ \lambda_{s_\kappa s_1} &\leftarrow \lambda_{s_\kappa s_1} - y \end{aligned}$$

Let λ^* denote the updated value. λ^* still satisfies equation (5). This implies $\lambda^* \in \ker_+(D\pi)$.

$$\sum_{\theta\theta'} \lambda_{\theta\theta'} \mathrm{vl}(\theta, \theta') - \sum_{\theta\theta'} \lambda^*_{\theta\theta'} \mathrm{vl}(\theta, \theta') = y \sum_{i=1}^{\kappa} \mathrm{vl}(s_i, s_{i+1}) \geq 0.$$

by cyclic monotonicity. So it suffices to show that for all $\lambda^* \in \ker_+(D\pi)$, we have $\lambda^* \cdot \mathrm{VL} \geq 0$.

Thus, we can assume that there is no cycle that $(s_1, s_2, \cdots, s_k)$ such that $\lambda_{s_1 s_2} \times \cdots \times \lambda_{s_\kappa s_1} \neq 0$. As $|\Theta| = 3$, there must be a θ_i such that

$$\lambda_{\theta_j \theta_i} = 0, \ \forall j \neq i.$$

We say that such i has the *lowest topological order*. And there must be a θ_i such that

$$\lambda_{\theta_i \theta_j} = 0, \ \forall j \neq i.$$

We say that such i has the *highest topological order*. We consider two cases.

Case 1: $K \geq 0$. The lowest topological order index i must be 1 or 2. By symmetry, we assume that it is 1. And the highest topological order index must be 3. Then by (5),

$$\begin{aligned} \lambda_{\theta_1 \theta_2} + \lambda_{\theta_1 \theta_3} &= Kt, \\ \lambda_{\theta_2 \theta_3} - \lambda_{\theta_1 \theta_2} &= K(1-t). \end{aligned} \tag{6}$$

We calculate $\text{VL} \cdot \lambda$,

$$\begin{aligned}
\sum_{\theta\theta'} \lambda_{\theta\theta'} \text{vl}(\theta, \theta') &= \lambda_{\theta_1\theta_2} \text{vl}(\theta_1, \theta_2) + \lambda_{\theta_1\theta_3} \text{vl}(\theta_1, \theta_3) + \lambda_{\theta_2\theta_3} \text{vl}(\theta_2, \theta_3) \\
&= \lambda_{\theta_1\theta_2} \text{vl}(\theta_1, \theta_2) + \lambda_{\theta_1\theta_3} [\pi(\theta_1) - \pi(\theta_3)] \cdot v(\theta_1) + \lambda_{\theta_2\theta_3} [\pi(\theta_2) - \pi(\theta_3)] \cdot v(\theta_2) \\
&= \lambda_{\theta_1\theta_2} \text{vl}(\theta_1, \theta_2) + (1-t)\lambda_{\theta_1\theta_3} \text{vl}(\theta_1, \theta_2) + t\lambda_{\theta_2\theta_3} \text{vl}(\theta_2, \theta_1) \\
&= Kt(1-t)(\text{vl}(\theta_1, \theta_2) + \text{vl}(\theta_2, \theta_1)) + t\lambda_{\theta_1\theta_2}(\text{vl}(\theta_1, \theta_2) + \text{vl}(\theta_2, \theta_1)) \\
&\geq 0
\end{aligned}$$

where the third equality follows by replacing $\pi(\theta_3)$ with $t\pi(\theta_1) + (1-t)\pi(\theta_2)$, and the last equality follows by (6).

Case 2: $K < 0$. The highest topological order index i must be 1 or 2. By symmetry, we assume that it is 1. And the lowest topological order index must be 3. Then by (5),

$$\lambda_{\theta_2\theta_1} + \lambda_{\theta_3\theta_1} = -Kt,$$

$$\lambda_{\theta_3\theta_2} - \lambda_{\theta_2\theta_1} = -K(1-t).$$

If $t = 0$, then $\lambda_{\theta_2\theta_1} = \lambda_{\theta_3\theta_1} = 0$ and $\text{vl}(\theta_3, \theta_2) = 0$, the value $\sum_{\theta\theta'} \lambda_{\theta\theta'} \text{vl}(\theta, \theta') = 0$. If $t > 0$, then

$$\begin{aligned}
\sum_{\theta\theta'} \lambda_{\theta\theta'} \text{vl}(\theta, \theta') &= \lambda_{\theta_3\theta_1} \text{vl}(\theta_3, \theta_1) + \lambda_{\theta_3\theta_2} \text{vl}(\theta_3, \theta_2) + \lambda_{\theta_2\theta_1} \text{vl}(\theta_2, \theta_1) \\
&= \frac{t-1}{t} \lambda_{\theta_3\theta_1} \text{vl}(\theta_3, \theta_2) + \lambda_{\theta_3\theta_2} \text{vl}(\theta_3, \theta_2) + \frac{1}{t} \lambda_{\theta_2\theta_1} \text{vl}(\theta_2, \theta_3) \\
&= \frac{\lambda_{\theta_2\theta_1}}{t} (\text{vl}(\theta_2, \theta_3) + \text{vl}(\theta_3, \theta_2)) \\
&\geq 0
\end{aligned}$$

Hence, we have $\text{VL} \in [\ker_+(D\pi)]^*$. By Theorem 1, the allocation rule is implementable with outcome-contingent transfers. □

Proof of Proposition 6. By Kantorovich Duality (Theorem 5.10 in Villani et al. (2009)), there exists $t : X \to \mathbb{R}$ such that for all for all θ and $x \in \textbf{supp}\{\pi(\cdot|\theta)\}$,

$$v(\theta, x) + t(x) \geq v(\theta, x') + t(x'), \forall x' \in X$$

if and only if $\lambda^* = \mu_0(\theta)\pi(x|\theta)$ is optimal solution for the following optimal transport problem,

$$\max_{\lambda\in\Delta(\Theta\times X)} \sum_{\theta,x} \lambda(\theta,x)v(\theta,x)$$
$$s.t. \lambda_\theta = \mu_0, \lambda_X = \nu$$

where μ_0 is a full-support distribution on Θ and $\nu(x) = \sum_{\theta\in\Theta} \mu(\theta)\pi(x|\theta)$. Again by Theorem 5.10 in Villani et al. (2009), λ^* is the solution of above optimal transport problem if and only if for any sequence $(\theta_1, x_1), \cdots, (\theta_m, x_m), (\theta_{m+1}, x_{m+1}) = (\theta_1, x_1)$ where $(\theta_i, x_i) \in \mathbf{supp}\{\lambda^*\}$,

$$\sum_{i=1}^{m} v(\theta_i, x_i) \geq \sum_{i=1}^{m} v(\theta_i, x_{i+1}).$$

Note that $(\theta_i, x_i) \in \mathbf{supp}\{\lambda^*\}$ if and only if $x_i \in \mathbf{supp}\{\pi(\cdot|\theta_i)\}$. □

Proof of Claim: All allocation rules are implementable with outcome-contingent transfers if and only if the agent's preference is additively separable.

The "if" part is taken care of by transfer $t(x) = -v_2(x)$. The "only if" part: Suppose all allocation rules are implementable with outcome-contingent transfers. Then we know that for any $\{\pi(\theta)\}_{\theta\in\Theta}$, by Observation 1, vl$(\cdot,\cdot)$ satisfies the cyclic monotonicity condition. Then for any $\theta \neq \theta' \in \Theta, x \neq x' \in X$. If we consider $\pi(x|\theta) = 1, \pi(x'|\theta') = 1$, the cyclic monotonicity condition requires that

$$v(\theta, x) + v(\theta', x') \geq v(\theta', x) + v(\theta, x').$$

If we consider $\pi(x'|\theta) = 1, \pi(x|\theta') = 1$, the cyclic monotonicity condition requires that

$$v(\theta, x) + v(\theta', x') \leq v(\theta', x) + v(\theta, x').$$

Then we know that for any $\theta \neq \theta' \in \Theta, x \neq x' \in X$, we must have $v(\theta, x) - v(\theta, x') = v(\theta', x) - v(\theta', x')$. Then fix $x_0 \in X$, then there is $v_2 \colon X \to \mathbb{R}$ such that $v(\theta, x) - v(\theta, x_0) = v_2(x)$ for all $\theta \in \Theta$. Thus we let $v_1(\theta) = v(\theta, x_0)$, then $v(\theta, x) = v_1(\theta) + v_2(x)$ which implies the agent's preference is additive separable. □

B Optimization over Allocation Rules

In this section, we take allocation rules as endogenous and consider a design problem. The planner's ex-post payoff function is $f(\theta, x, t)$. The planner sets up an outcome-contingent allocation and transfer rule $(\pi(\theta), T)$ to maximize expected payoff

$$E_\theta\{\sum_x \pi(x|\theta) f(x, \theta, t(x))\}$$

subject to the IC constraint

$$\forall \theta, \theta' \in \Theta, \quad \pi(\theta) \cdot (v(\theta) + T) \geq \pi(\theta') \cdot (v(\theta) + T)$$

and IR (participation) constraint

$$\forall \theta \in \Theta, \quad \pi(\theta) \cdot (v(\theta) + T) \geq 0.$$

We show that it is without loss to restrict attention to convex independent allocation rules.

Proposition 7. *It is without loss for the planner to focus on convex independent allocation rules.*

Proof. Suppose that (π, T) satisfies the IC and IR constraints. We show that there is a convex independent allocation rule π' such that (π', T) yields a weakly larger payoff for the planner.

Let $\Theta' \subset \Theta$ collect all θ such that $\pi(\theta)$ is the extreme point of the convex hull of $\{\pi(\theta)|\theta \in \Theta\}$. Fix some $\theta_i \in \Theta$. There exists $\lambda(\cdot) : \Theta' \to \mathbb{R}_{\geq 0}$ such that

$$\pi(\theta_i) = \sum_{\theta \in \Theta'} \lambda(\theta)\pi(\theta) \quad \text{and} \quad \sum_{\theta \in \Theta'} \lambda(\theta) = 1.$$

Note that the planner's expected payoff conditional on θ_i is

$$\sum_x \pi(x|\theta_i) f(x, \theta_i, t(x)) = \sum_{\theta \in \Theta'} \lambda(\theta) \sum_x \pi(x|\theta) f(x, \theta_i, t(x)).$$

There must exist some $\theta_i' \in \Theta'$ such that

$$\sum_x \pi(x|\theta_i') f(x, \theta_i, t(x)) \geq \sum_x \pi(x|\theta_i) f(x, \theta_i, t(x)).$$

We define a new allocation rule π' by $\pi'(\theta_i) = \pi(\theta_i')$. Note that (π', T) generates a weakly higher payoff for the planner. Lastly, by Proposition 4, agent θ_i's payoff does not change,

$$(v(\theta_i) + T) \cdot \pi(\theta_i) = (v(\theta_i) + T) \cdot \pi'(\theta_i).$$

Thus, IR still holds. The set of IC constraints is smaller due to

$$\{\pi'(\theta)|\theta \in \Theta\} = \{\pi(\theta)|\theta \in \Theta'\} \subset \{\pi(\theta)|\theta \in \Theta\}$$

Thus, IC still holds.

□